Lecture 17

Maxima and Minima

Objectives

In the coming lectures, we will learn how to use **the first and second derivatives** to explore the behavior of a function, namely how to find

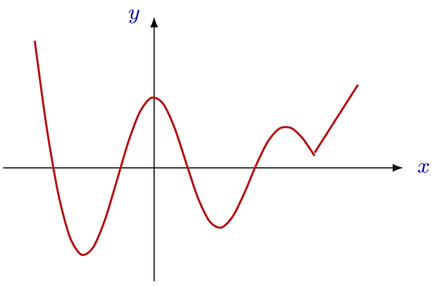
- maxima and minima
- intervals where the function is increasing and intervals where it is decreasing
- intervals of concavity
- inflection points.

We will see how to use the obtained information to draw the **graph** of the function.

This lecture is devoted to finding **maxima and minima** of functions.

What can we see on the graph of a function?

Let us have a look on the graph of a function:

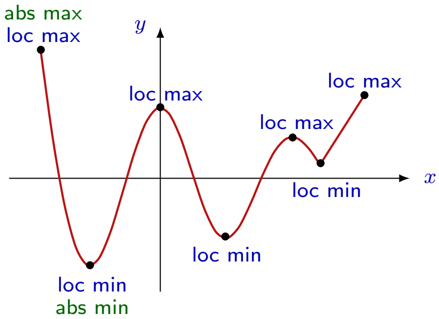


Which **characteristic features** of a function can we see on its graph?
Humps and valleys, corners, uphills and downhills, concavities, highest and lowest points.

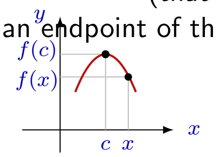
These features provide essential information about a function and may be expressed in terms of the **derivatives** of the function.

Extracting information about a function from its derivatives is an essential part of the analysis of functions.

Maxima



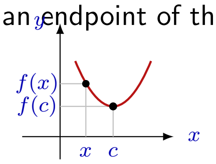
Definition. A function f has a *local maximum value* $f(c)$ at the point c
if $f(x) \leq f(c)$ for all x in the domain near c
(that is, for all $x \in (c-\delta, c+\delta)$ for a sufficiently small $\delta > 0$) when c is not an endpoint of the domain.



In this case, near $x = c$, the graph of $y = f(x)$ is located below or on the horizontal line $y = f(c)$.

Minima, local and absolute extrema

Definition. A function f has a *local minimum value* $f(c)$ at the point c
if $f(x) \geq f(c)$ for all x near c
(that is, for all $x \in (c-\delta, c+\delta)$ for a sufficiently small $\delta > 0$) when c is not an endpoint of the domain.



In this case, near $x = c$, the graph of $y = f(x)$ is located above or on the horizontal line $y = f(c)$.

The local maximum and minimum values are called *local extreme values*.

Definition. A function f has an *absolute maximum value* $f(c)$ at the point c
if $f(c) \geq f(x)$ for all x in the domain.

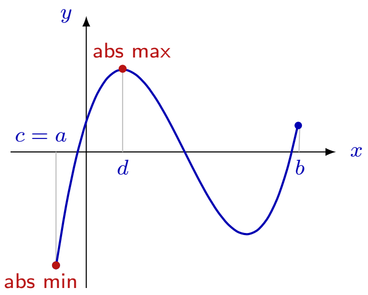
A function f has an *absolute minimum value* $f(c)$ at the point c
if $f(c) \leq f(x)$ for all x in the domain.

The absolute maximum and minimum values are called the *extreme values*.

The Extreme Value theorem (Max-Min theorem)

Theorem. If f is **continuous** on a **closed** interval $[a, b]$,
then f attains its extreme values on this interval.
That is, there are points c and d in $[a, b]$, such that

$f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.



Remarks: **1.** An extreme value may be taken more than once.
2. The Extreme Value Theorem is fundamental in the analysis of functions.
3. This theorem gets proved in a course on Mathematical Analysis.

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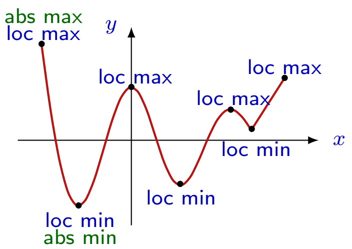
Where are the extreme values of a function located?

Finding the **extreme values** of a function
is one of the most important tasks of calculus and its applications.

Let f be a continuous function. Where can its extrema be found?



Our next goal is to prove the following:

☞ A function may have extreme values only at points of three special types:

- **critical points** of f (points x where $f'(x) = 0$),
- **singular points** of f (points x where $f'(x)$ doesn't exist),
- **endpoints** of the domain of f.

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Fermat's theorem (Interior Extremum theorem)

Theorem. Let f be a function defined on (a,b). If f has a local extremum at $x \in (a,b)$ and f is differentiable at x, then $f'(x)=0$, that is, x is a **critical point** of f.

Proof. Suppose that f has a local maximum value at x. This means that for all sufficiently small h with $x+h \in (a,b)$, we have $f(x+h) \le f(x)$, that is $f(x+h)-f(x) \le 0$. Note that h can be positive or negative.

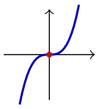
If $h>0$, then $\dfrac{f(x+h)-f(x)}{h} \le 0$ and $\lim\limits_{h\to 0^+} \dfrac{f(x+h)-f(x)}{h} \le 0$.

If $h<0$, then $\dfrac{f(x+h)-f(x)}{h} \ge 0$ and $\lim\limits_{h\to 0^-} \dfrac{f(x+h)-f(x)}{h} \ge 0$.

Since f is differentiable at x, these two limits must both be equal to $f'(x)$.
That means that $f'(x) \le 0$ and $f'(x) \ge 0$. Hence $f'(x)=0$.
The proof for a local **minimum** is similar. □

⚠ **Warning:** The converse of the theorem is **not** true.
That is, it is **not** true that if $f'(x)=0$ then x is an extreme point.

For example, $f(x)=x^3$ has vanishing derivative at 0 ($f'(0)=3x^2\big|_{x=0}=0$),
but f has neither a local maximum nor a local minimum at $x=0$.



Locating the extreme values

Theorem. If a function f is defined on an interval I
and has a local extremum at a point $x_0 \in I$, then x_0 must be either

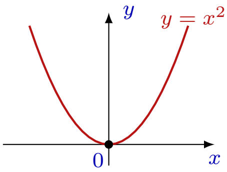
- a **critical point** of f (where $f'(x_0)=0$),
- a **singular point** of f (where $f'(x_0)$ doesn't exist) or
- an **endpoint** of I.

Proof. Suppose that f has a local extremum at x_0
and that x_0 is neither a singular point of f nor an endpoint of I.
Since x_0 is not a singular point, then f is differentiable at x_0.

Since x_0 is not an endpoint of I, we may apply Fermat's theorem
and obtain $f'(x_0)=0$. This means that x_0 is a critical point of f. □

Extreme values: examples

Example 1. Look at the graph of $f(x) = x^2$:

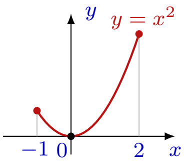


At $x = 0$, f has a local minimum
(which is also the absolute minimum).

$x = 0$ is a **critical point** of f :

$f'(x) = (x^2)'\Big|_{x=0} = (2x)\Big|_{x=0} = 0$.

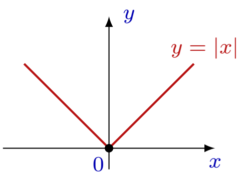
Example 2. Consider the function $f(x) = x^2$ restricted to $[-1, 2]$:



f has a local minimum at $x = 0$,
which is a **critical point** of f ,
and local maximums at $x = -1$ and $x = 2$,
which are **endpoints** of the domain.
f has the absolute minimum at $x = 0$ (**critical point**),
and the absolute maximum at $x = 2$ (**endpoint**).

Extreme values: examples

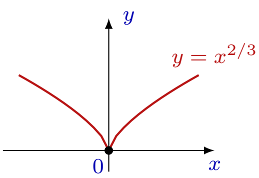
Example 3. Consider $f(x) = |x|$.



At $x = 0$, f has a local minimum
which is also the absolute minimum.

$x = 0$ is a **singular point** of f : $f'(0)$ does not exist.

Example 4. Consider $f(x) = x^{2/3}$.



At $x = 0$, f has a local minimum
which is also the absolute minimum.

$x = 0$ is a **singular point** of f .

Indeed, $f'(x) = (x^{2/3})' = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$

and $f'(0)$ does not exist.

Extreme values on a closed interval

Problem 1. Find the maximum and minimum values of the function
$f(x) = 2x^3 + 3x^2 - 12x$ on the interval $[-1, 2]$.

Solution. The function f is **continuous**, and, by the Extreme Value Theorem, attains maximum and minimum values on the **closed** interval $[-1, 2]$.

We search for the extreme values
among the critical and singular points of f, and the endpoints of the interval.

To find critical points, we calculate the derivative

$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x - 1)(x + 2)$

and solve the equation $f'(x) = 0 \iff x = 1$ or $x = -2$.

There are two **critical points**: $x = 1$, $x = -2$. Only $x = 1$ belongs to the interval $[-1, 2]$.

f has **no** singular points, since f' exists for all x.

There are two **endpoints**: $x = -1$ and $x = 2$.

Extreme values on a closed interval

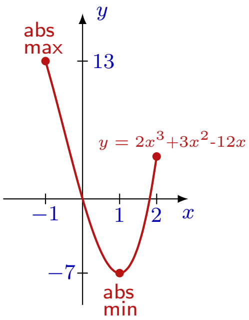
We calculate the values of $f(x) = 2x^3 + 3x^2 - 12x$ at the **critical point** of f, which is $x = 1$, and at the **endpoints** of the interval, which are $x = -1$ and $x = 2$:

$f(1) = 2 \cdot 1^3 + 3 \cdot 1^2 - 12 \cdot 1 = \boxed{-7}$,
$f(-1) = 2(-1)^3 + 3(-1)^2 - 12(-1) = \boxed{13}$,
$f(2) = 2 \cdot 2^3 + 3 \cdot 2^2 - 12 \cdot 2 = \boxed{4}$.

Choose the maximal (highest) and the minimal (lowest) values among them.
The answer to the problem is

$\max\limits_{[-1,2]} f = 13$, attained at the endpoint $x = -1$,

$\min\limits_{[-1,2]} f = -7$, attained at the critical point $x = 1$.

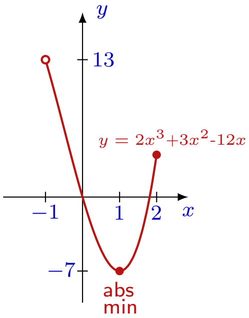


What if the interval is not closed?

In this case, a function may or may not have maximum/minimum on the interval.

Example 1. Does the function $f(x) = 2x^3 + 3x^2 - 12x$ have a maximum or a minimum on the interval $(-1, 2]$?

Solution.

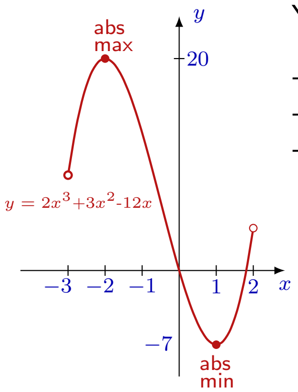


As we see from the graph,
f does not reach an absolute maximum on $(-1, 2]$.
The absolute minimum of -7 is attained at $x = 1$.

What if the interval is not closed?

Example 2. Does the function $f(x) = 2x^3 + 3x^2 - 12x$ have a maximum or a minimum on the interval $(-3, 2)$?

Solution.



Yes, it does, although the interval $(-3, 2)$ is **not** closed.

The maximal value of f is 20 . It is attained at $x = -2$.
The minimal value of f is -7 . It is attained at $x = 1$.
The points $x = -2$ and $x = 1$ are the critical points of f .

We see that **graphs** are very useful for understanding the behavior of functions.
To draw graphs accurately, we need to develop the analysis of functions further.

Summary

In this lecture we learned

• the Extreme Value Theorem

• how to find the extreme values of a continuous function on a closed interval.

Comprehension checkpoint

• Assume that a function has a local minimum at a point.
What can you say about the function at this point?

• Draw the graph of a function defined on $[-3, 4]$ that has

local maxima at $x = -3, 2, 4,$
local minima at $x = 1, 3,$
critical points at $x = 1, 2$
a singular point at $x = 3,$ such that
the absolute maximum of 2 is attained at $x = -3,$
the absolute minimum of -1 is attained at $x = 3.$