Lecture 16

# Linearization

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## Objectives

In the coming lectures, we will study **applications** of the derivative:

- Linear approximation
- Analysis of functions
- Implicit differentiation
- Limits of indeterminate forms (l'Hôpital's rule)
- Related rates problems
- Optimization problems

In this lecture, we will discuss

- Linear approximation of functions and its applications.



## Linear approximation

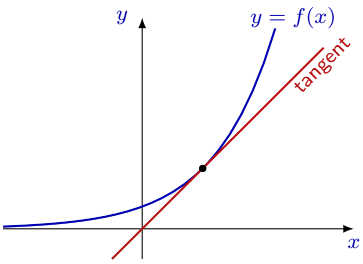
In applications, we may encounter difficulties in finding **exact** solutions to the problems.
Often **approximate** solutions are acceptable within some tolerance.
The **simplest** approximation of a function is given by a linear function.

In this section, we will study how a differentiable function may be approximated by a **linear function**.

We have already seen that the tangent line goes **very close** to the graph of the function.

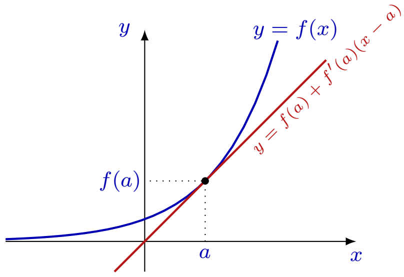
The tangent line describes the behavior of the function near the point of tangency **better** than any other line.

It makes sense to use the tangent line as a **linear approximation** to the graph.



## Linearization

Let $f$ be a function differentiable at the point $x = a$. The equation of the tangent line to the graph of $f$ at the point $x = a$ is $y = f(a) + f'(a)(x - a)$.




**Definition.** The *linearization*, or *linear approximation*, of the function $f$ near point $x = a$ is the linear function $L(x) = f(a) + f'(a)(x - a)$.

$f(x) \approx L(x)$ near $x = a$.



## Examples of linearization

**Example 1.** Find the linear approximation to $f(x) = \sin x$ near $x = 0$.

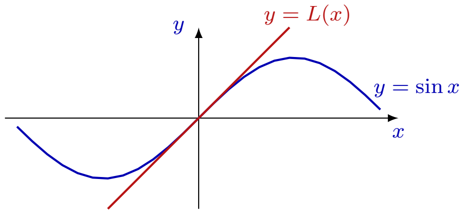
**Solution.** The linear approximation is the function $L(x) = f(a) + f'(a)(x - a)$, where $f(x) = \sin x$ and $a = 0$.

Since $f'(0) = f'(x)\Big|_{x=0} = (\sin x)'\Big|_{x=0} = \cos x\Big|_{x=0} = \cos 0 = 1$ and

$f(0) = \sin 0 = 0$, we find

$L(x) = 0 + 1 \cdot (x - 0) \iff L(x) = x$.

The linear function approximating $f(x) = \sin x$ near $x = 0$ is $L(x) = x$:




We write $\sin x \approx x$ for small $x$.

## Examples of linearizations

**Example 2.** Find the linear approximations to $f(x) = \sqrt{x}$ near $x = 1$ and $x = 4$.

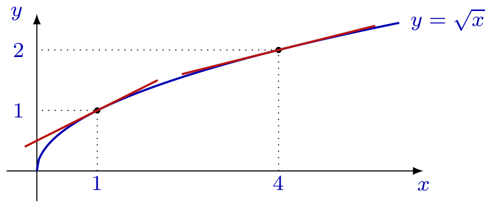
**Solution.** $f'(x) = (\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$. So $f'(1) = \dfrac{1}{2}$, $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

The linearization near $x = 1$ is

$$L(x) = f(1) + f'(1)(x-1) \iff L(x) = 1 + \frac{1}{2}(x-1) \iff L(x) = \frac{x}{2} + \frac{1}{2}.$$

The linearization near $x = 4$ is

$$L(x) = f(4) + f'(4)(x-4) \iff L(x) = 2 + \frac{1}{4}(x-4) \iff L(x) = \frac{x}{4} + 1.$$






## Approximate calculations

**Example 3.** Use linearization to find an approximate value for $\dfrac{1}{1.99}$.

Is this approximation an overestimate or an underestimate?

**Solution.** We can easily find the value of $\dfrac{1}{2}$, and since $1.99$ is near $2$,

we may use the linear approximation $L(x) = f(a) + f'(a)(x-a)$ for $f(x) = \dfrac{1}{x}$ and $a = 2$.

Do the math:

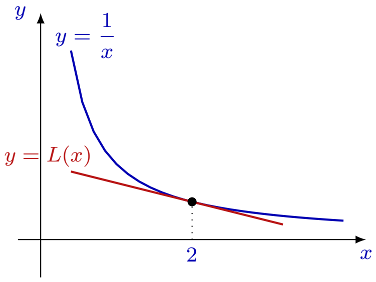
$$f'(2) = f'(x)\Big|_{x=2} = \left(\frac{1}{x}\right)'\Big|_{x=2} = -\frac{1}{x^2}\Big|_{x=2} = -\frac{1}{4}.$$ Since $f(2) = \dfrac{1}{2}$, we have

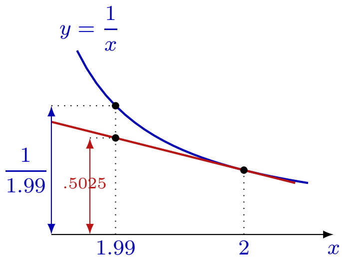
$L(x) = \dfrac{1}{2} - \dfrac{1}{4}(x-2)$. Leave this formula as is, **without** simplifications.

$$\frac{1}{1.99} = f(1.99) \approx L(1.99) = \frac{1}{2} - \frac{1}{4}(1.99-2) = 0.5 + \frac{0.01}{4} = 0.5 + 0.0025 = \boxed{0.5025}$$

## Approximate calculations

Now we have to answer the question:
whether the obtained approximation is an **overestimate** or an **underestimate**, that is,
Is the approximation $0.5025$ **greater** than or **less** than the true value of $\dfrac{1}{1.99}$ ?




The tangent line is **below** the graph of the function,
therefore, the approximate value, $0.5025$, is **less** than the actual value of $\dfrac{1}{1.99}$.

By calculator: $1/1.99 = 0.502512...$ Linearization gave four correct decimals!



## Approximation of functions

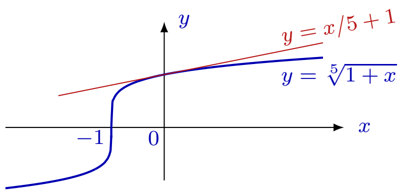
**Example.** Show that $\sqrt[5]{1+x} \approx 1 + \dfrac{x}{5}$ for small $x$.

**Solution.** Consider the function $f(x) = \sqrt[5]{1+x}$. It is differentiable at $x = 0$,
therefore, by linearization,

$f(x) \approx L(x) = f(0) + f'(0)(x - 0)$ for $x$ near $0$.

Since $f'(x) = \dfrac{d}{dx}(1+x)^{1/5} = \dfrac{1}{5}(1+x)^{-4/5}$, we have $f'(0) = \dfrac{1}{5}$ and

$f(x) \approx L(x) = f(0) + f'(0)(x - 0) = 1 + \dfrac{1}{5}x = 1 + \dfrac{x}{5}$, as required.



## Linearization in terms of differentials

Let $y = f(x)$ be a function differentiable at the point $x = a$.
According to the **linearization** formula,

$f(x) \approx f(a) + f'(a)(x - a)$ for all $x$ near $a$, or, equivalently,

$f(x) - f(a) \approx f'(a)(x - a)$. Let $\Delta x = x - a$. Then

$f(a + \Delta x) - f(a) \approx f'(a)\Delta x$. Rewrite this formula in terms of $x$ instead of $a$:

$f(x + \Delta x) - f(x) \approx f'(x)\Delta x$. Let $\Delta y = f(x + \Delta x) - f(x)$. Then

$\Delta y \approx f'(x)\Delta x \iff \Delta y \approx \dfrac{dy}{dx}\Delta x.$

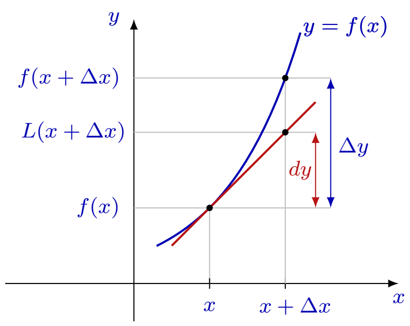
Define the *differential* of the function as $dy = \dfrac{dy}{dx}\Delta x.$ Then

$$\Delta y \approx \frac{dy}{dx}\Delta x \iff \boxed{\Delta y \approx dy}$$

the increment (change) of the function $\approx$ the differential of the function.



## $\Delta y$ and $dy$




$\boxed{\Delta y \approx dy}$

$\Delta y = f(x + \Delta x) - f(x), \;\; dy = f'(x)\Delta x$

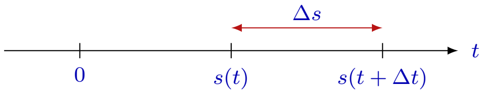
The smaller $\Delta x$ is, the better the approximation $\Delta y \approx dy$.

## Calculations with differentials

**Example 1.** A point moves along a straight line according the law $s(t) = 5t^2$, where $t$ is time in seconds and $s(t)$ is the distance from the origin, in meters.

At time moment $t = 2$ sec, calculate the displacement $\Delta s$ and the differential $ds$ over the time intervals **a)** $\Delta t = 1$ sec **b)** $\Delta t = 0.1$ sec.

**Solution.**




The displacement is $\Delta s = s(t + \Delta t) - s(t)$. It depends on $t$ and $\Delta t$.
The differential is $ds = s'(t)\Delta t$. It also depends on $t$ and $\Delta t$.
**a)** For $t = 2$ and $\Delta t = 1$, $\Delta s = s(2 + 1) - s(2) = s(3) - s(2) = 5 \cdot 3^2 - 5 \cdot 2^2 = 5(9 - 4) = 25$ (m)
$ds = 10t\Delta t = 10 \cdot 2 \cdot 1 = 20$ (m)
**b)** For $t = 2$ and $\Delta t = 0.1$, $\Delta s = s(2 + 0.1) - s(2) = s(2.1) - s(2) = 5 \cdot (2.1)^2 - 5 \cdot 2^2 = 5(4.41 - 4) = 2.05$ (m)
$ds = 10t\Delta t = 10 \cdot 2 \cdot 0.1 = 2$ (m). By linearization, $\Delta s \approx ds$, so for $\Delta t = 1$, $25 \approx 20$ and for $\Delta t = 0.1$, $2.05 \approx 2$.

☞ The smaller $\Delta t$ is, the better the approximation.



## Calculations with differentials

**Example 2.** A spherical balloon inflates so that its radius increases from $5$ cm to $5.4$ cm. By approximately how much does the volume increase?

**Solution.** The volume $V$ of a ball of radius $r$ is $V = \frac{4}{3}\pi r^3$.

The increase of volume from $r = 5$ to $r + \Delta r = 5.4$ ($\Delta r = 0.4$) is

$\Delta V = V(r + \Delta r) - V(r) = V(5.4) - V(5)$.

By linearization, $\Delta V \approx dV = V'(r)\Delta r = 4\pi r^2 \Delta r$.

When $r = 5$ and $\Delta r = 0.4$, we get

$\Delta V \approx 4\pi \cdot 5^2 \cdot 0.4 = 40\pi \approx 125.7$ cm$^3$.

## Coulomb's law

According to the **Coulomb's law**, the electrostatic force $F$ between two charges $q_1$ and $q_2$ located at a distance $r$ from each other, is given by $F = k\frac{q_1 q_2}{r^2}$, where $k$ is Coulomb's constant.

If the distance between the charges was measured to be $1m$ with an error of at most $1cm$, what is the **relative error** in the calculation of the electrostatic force?

**Solution.** The relative error is $\left|\frac{\Delta F}{F}\right|$.

By linearization, $\Delta F \approx dF = -2k\frac{q_1 q_2}{r^3}\Delta r$, therefore

$$\left|\frac{\Delta F}{F}\right| \approx \left|\frac{-2k\frac{q_1 q_2}{r^3}\Delta r}{k\frac{q_1 q_2}{r^2}}\right| = \frac{2\Delta r}{r}.$$

We are given $r = 1m$ and $\Delta r = 1cm = 0.01m$, so

$$\left|\frac{\Delta F}{F}\right| \approx \frac{2\Delta r}{r} = \frac{2 \cdot 0.01}{1} = 0.02 = 2\%$$



## Summary

In this lecture we studied **linear approximation** of functions.

Remember:

- A function $y = f(x)$ is approximated near $x = a$ by a linear function $L(x) = f(a) + f'(a)(x - a)$.
- An increment $\Delta y$ of a function $y = f(x)$ is approximated by the differential of the function, $dy = f'(x)\Delta x$, namely, $\Delta y \approx dy$.

**Comprehension checkpoint**

• Explain why $\tan x \approx x$ for small $x$.

• Let $y = f(x)$ be a differentiable function. Explain what are $dy$ and $\Delta y$.
Draw a picture!

• Use linearizarion to find an approximate value of $\sqrt[3]{8.03}$.
Give geometric interpretation of your calculations.

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