Lecture 15

Derivatives of Inverse Functions

Objectives

In this lecture, we

- establish the rule for differentiation of an **inverse** function,
- compute derivatives of the **logarithmic** and **power** functions and **inverse** trigonometric functions.
- learn the technique of **logarithmic differentiation**

Derivatives of inverse functions

Theorem. Let $y = f(x)$ be an invertible function and $x = f^{-1}(y)$ be its inverse.

If f is differentiable at x and $f'(x) \neq 0$,

then f^{-1} is differentiable at $y = f(x)$ and $\boxed{(f^{-1})'(y) = \dfrac{1}{f'(x)}}$

Proof. By the definition of the inverse, $f^{-1}(f(x)) = x$.
Differentiate this identity using the chain rule:

$$\frac{d}{dx} f^{-1}(f(x)) = \frac{d}{dx} x \iff \frac{df^{-1}}{dy}(f(x)) \cdot \frac{df}{dx}(x) = 1$$

$$\iff \frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dx}(x)} \iff (f^{-1})'(y) = \frac{1}{f'(x)}.$$

Notice that the latter identity makes sense since $f'(x) \neq 0$. □

The derivative of the natural logarithm

For the exponential function $f(x) = e^x$ (or $y = e^x$), the inverse is $f^{-1}(y) = \ln y$ (by definition of $\ln$).

Since $f'(x) = (e^x)' = e^x \neq 0$ for all x,

then $f^{-1}(y) = \ln y$ is differentiable where it is defined, that is, for all $y > 0$.

By the theorem about derivatives of inverse functions,

$(f^{-1})'(y) = \dfrac{1}{f'(x)}$. In our case, this formula gives

$(\ln y)' = \dfrac{1}{e^x}$. Since $x = \ln y$, we get

$(\ln y)' = \dfrac{1}{e^{\ln y}} \iff (\ln y)' = \dfrac{1}{y}$.

Switching to the variable x, we get $\boxed{(\ln x)' = \dfrac{1}{x}}$

Derivatives of logarithmic functions

Theorem. $\boxed{(\log_a x)' = \dfrac{1}{x \ln a}}$

Proof. By the base change property of logarithms, $\log_a x = \dfrac{\ln x}{\ln a}$. Therefore,

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}. \qquad \square$$

Example. Find the derivatives of the functions $y = \ln(\sin x)$ and $y = \sqrt{\log_2 x}$.

Solution. $\dfrac{d}{dx}\ln(\sin x) = \dfrac{1}{\sin x} \cdot (\sin x)' = \dfrac{1}{\sin x} \cdot \cos x = \cot x$.

$$\frac{d}{dx}\sqrt{\log_2 x} = \frac{d}{dx}(\log_2 x)^{1/2} = \frac{1}{2}(\log_2 x)^{-1/2} \cdot (\log_2 x)' = \frac{1}{2\sqrt{\log_2 x}} \cdot \frac{1}{x \ln 2}$$

$$= \frac{1}{(2\ln 2)x\sqrt{\log_2 x}}.$$

Derivatives of power functions (proof)

Theorem. $\boxed{(x^a)' = ax^{a-1}}$ for any $a \in \mathbb{R}$.

Proof. $x^a = e^{\ln x^a} = e^{a \ln x}$.

$$(x^a)' = (e^{a\ln x})' = e^{a\ln x} \cdot (a \ln x)' = e^{a \ln x} \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = ax^{a-1}\,. \qquad \square$$

Example. Differentiate the following functions: $y = x^e$, $y = (x + \sin x)^3$.

Solution. $(x^e)' = ex^{e-1}$

$$\frac{d}{dx}(x + \sin x)^3 = 3(x + \sin x)^2 \cdot \frac{d}{dx}(x + \sin x) = 3(x + \sin x)^2(1 + \cos x).$$

The number e as a limit

Theorem. $\lim\limits_{x\to 0}(1 + x)^{1/x} = e$.

Proof. $(1 + x)^{1/x} = e^{\ln((1+x)^{1/x})} = e^{\frac{1}{x}\ln(1+x)}$.

$$\frac{\ln(1 + x)}{x} = \frac{\ln(1 + x) - \ln 1}{x} \underset{x\to 0}{\longrightarrow} \frac{d}{dx}(\ln x)\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1\,.$$

So $\frac{1}{x}\ln(1 + x) \underset{x\to 0}{\longrightarrow} 1$, therefore,

$e^{\frac{1}{x}\ln(1+x)} \underset{x\to 0}{\longrightarrow} e^1 = e$, that is $\boxed{\lim\limits_{x\to 0}(1 + x)^{1/x} = e}$

Corollary. $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$.

Proof. $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim\limits_{t\to 0}(1 + t)^{1/t} = e.$

The derivative of the arcsine

Theorem. $\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$ for $x \in (-1,1)$

Proof. For the function $f(x) = \sin x$, where $x \in [-\pi/2, \pi/2]$,

the inverse is $f^{-1}(y) = \arcsin y$, where $y \in [-1,1]$.

By the theorem about derivatives of inverse functions,

$(f^{-1})'(y) = \frac{1}{f'(x)}$ for all points where $f'(x) \neq 0$.

In our case, this formula gives $(\arcsin y)' = 1/(\sin x)' = 1/\cos x = ?$

$\cos x = \sqrt{1-\sin^2 x}$ since $\cos x \geq 0$ for $x \in [-\pi/2, \pi/2]$

$= \sqrt{1-y^2}$ since $\sin x = y$. Therefore, $(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}$.

The formula is valid for $y \in (-1,1)$, that is where $f'(x) = \cos x \neq 0$.

Switching to x gives $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, where $x \in (-1,1)$.

The derivative of the arccosine

Lemma. $\arcsin y + \arccos y = \frac{\pi}{2}$ for $y \in [-1,1]$.

Proof. Let $y = \sin x$. Since $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, we get $y = \cos\left(\frac{\pi}{2} - x\right)$, or $\arccos y = \frac{\pi}{2} - x$.

Therefore, $\arccos y = \frac{\pi}{2} - \arcsin y$, that is,

$\arcsin y + \arccos y = \frac{\pi}{2}$. $\square$

Theorem. $\boxed{(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}$ for $x \in (-1,1)$

Proof. $\arcsin x + \arccos x = \frac{\pi}{2} \implies \arccos x = \frac{\pi}{2} - \arcsin x$. Therefore,

$(\arccos x)' = \left(\frac{\pi}{2} - \arcsin x\right)' = -(\arcsin x)' = -\frac{1}{\sqrt{1-x^2}}$. $\square$

The derivative of the arctangent

Theorem. $\boxed{(\arctan x)' = \dfrac{1}{1+x^2}}$ for any $x \in \mathbb{R}$.

Proof. For the function $f(x) = \tan x$, where $x \in (-\pi/2, \pi/2)$,

the inverse is $f^{-1}(y) = \arctan y$, where $y \in \mathbb{R}$.

By the theorem about derivatives of inverse functions,

$(f^{-1})'(y) = \dfrac{1}{f'(x)}$. In our case, this formula gives

$(\arctan y)' = \dfrac{1}{(\tan x)'} = \cos^2 x = ?$

$1 + \tan^2 x = 1 + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$. So $\cos^2 x = \dfrac{1}{1+\tan^2 x} = \dfrac{1}{1+y^2}$

and $(\arctan y)' = \cos^2 x = \dfrac{1}{1+y^2}$. Switch to x: $(\arctan x)' = \dfrac{1}{1+x^2}$. $\square$

Table of derivatives

$f(x)$	$f'(x)$
C	0
x^a	ax^{a-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\dfrac{1}{x}$
$\log_a x$	$\dfrac{1}{x \ln a}$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$
$\cot x$	$-\dfrac{1}{\sin^2 x}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

How to differentiate x^x

Problem. Find the derivative of $y = x^x$.

Solution. The function is neither a power nor an exponential.

It seems that there is no rule for differentiating it.

To get around this difficulty, let us take the natural logarithm of both sides of the equation $y = x^x$:

$\ln y = \ln x^x \iff \ln y = x \ln x$. Differentiate both sides of this last equation.
Keep in mind that $y = y(x)$ is a function of x ,

and so $\ln y$ should be differentiated according the **chain rule**:

$$\frac{d}{dx} \ln y(x) = \frac{d}{dx}(x \ln x) \iff \frac{1}{y(x)} \cdot y'(x) = (x)' \ln x + x \cdot (\ln x)'$$

$$\iff \frac{y'(x)}{y(x)} = \ln x + x \cdot \frac{1}{x} \iff \frac{y'(x)}{y(x)} = \ln x + 1 \quad \left(\frac{y'(x)}{y(x)} \text{ is called the } \textit{logarithmic derivative} \text{ of } y \right)$$

$$\iff y'(x) = (\ln x + 1) y(x) \iff y'(x) = (\ln x + 1) x^x \text{. Therefore, } \frac{d}{dx}(x^x) = (\ln x + 1) x^x \text{.}$$

Logarithmic differentiation

The technique used in differentiation of $y = x^x$ is called *logarithmic differentiation*.

It is used to differentiate complicated functions that involve products, quotients, or powers.

Example 1. Differentiate the function $y = \dfrac{\sqrt{x}(x^2+3)}{x^4(2x^3 - x)}$.

Solution. Using the quotient rule may cause a headache.
Let us get around that with **logarithmic differentiation**.

$$\ln y = \ln \frac{\sqrt{x}(x^2+3)}{x^4(2x^3 - x)} \qquad \left(\ln \frac{ab}{cd} = \ln a + \ln b - \ln c - \ln d \right)$$

$$= \ln \sqrt{x} + \ln(x^2+3) - \ln x^4 - \ln(2x^3 - x) = \frac{1}{2} \ln x + \ln(x^2+3) - 4 \ln x - \ln(2x^3 - x)$$

Logarithmic differentiation

For $y = \dfrac{\sqrt{x}(x^2+3)}{x^4(2x^3-x)}$ we have found that

$\ln y = \dfrac{1}{2}\ln x + \ln(x^2+3) - 4\ln x - \ln(2x^3-x)$. Now it's time to differentiate:

$\dfrac{y'}{y} = \dfrac{1}{2x} + \dfrac{2x}{x^2+3} - \dfrac{4}{x} - \dfrac{6x^2-1}{2x^3-x}$. Therefore,

$y' = y\left(\dfrac{1}{2x} + \dfrac{2x}{x^2+3} - \dfrac{4}{x} - \dfrac{6x^2-1}{2x^3-x}\right)$, that is,

$y' = \dfrac{\sqrt{x}(x^2+3)}{x^4(2x^3-x)}\left(\dfrac{1}{2x} + \dfrac{2x}{x^2+3} - \dfrac{4}{x} - \dfrac{6x^2-1}{2x^3-x}\right)$. Simplify the right hand side:

$y' = \dfrac{(-18x^4-73x^2+27)\sqrt{x}}{2x^6(2x^2-1)^2}$.

Logarithmic differentiation

Example 2. Find the incline angle of the tangent line to the graph of the function $y = \sqrt[3]{(x+1)(x^2+1)(x^3+1)}$ at the point $(1,2)$.

Solution. Let θ be the incline angle of the tangent line at $(1,2)$. Then $\tan\theta = y'(1)$ (it is the slope of the tangent line at $x=1$).

Use **logarithmic differentiation** to find y':

$$\ln y = \ln \sqrt[3]{(x+1)(x^2+1)(x^3+1)} = \ln((x+1)(x^2+1)(x^3+1))^{1/3}$$
$$= \frac{1}{3}\Big(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\Big).$$

$\dfrac{y'}{y} = \dfrac{1}{3}\left(\dfrac{1}{x+1} + \dfrac{2x}{x^2+1} + \dfrac{3x^2}{x^3+1}\right)$. Substitute $x=1$:

$\dfrac{y'(1)}{y(1)} = \dfrac{1}{3}\left(\dfrac{1}{2} + \dfrac{2}{2} + \dfrac{3}{2}\right)$. Therefore, since $y(1) = \sqrt[3]{2\cdot 2\cdot 2} = 2$, this gives

$y'(1) = 2\cdot\dfrac{1}{3}\cdot 3 = 2$. Therefore, $\tan\theta = 2$ and $\theta = \arctan 2$.

Summary

In this lecture, we learned

- how the derivative of a function and its inverse are related
- how to calculate derivatives of
  logarithmic functions, power functions, and the inverse trigonometric functions
- how to use the technique of **logarithmic differentiation**.

Comprehension checkpoint

- Fill in the tables:

f	C	x^a	a^x	e^x	$\ln x$	$\log_a x$
f'						

f	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\arcsin x$	$\arccos x$	$\arctan x$
f'							

- Use logarithmic differentiation to find the derivatives of the following functions:

$$y = \frac{3x^2 - 1}{(x^5 + x - 2)(x + 2)}, \quad y = (\sin x)^{\cos x}.$$