Lecture 14

Derivatives of Trigonometric Functions

Objectives

In this lecture, we

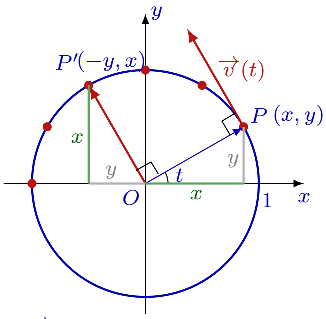
compute derivatives of the **trigonometric** functions.

Our computation will use **vectors**.

Derivatives of sine and cosine

Theorem. $\frac{d}{dt}\sin t = \cos t, \; \frac{d}{dt}\cos t = -\sin t$.

Proof. Consider the unit circle and a point P moving along it with unit speed.



$\overrightarrow{OP}$ is the position vector of P. Its coordinates depend on the angle t it makes with the x-axis:

$\overrightarrow{OP} = (x(t), y(t))$. By the definition of cosine and sine, $x(t) = \cos t, \; y(t) = \sin t$.

The velocity $\overrightarrow{v}(t) = (x'(t), y'(t))$ is tangent to the circle.

So, $\overrightarrow{v}(t) \perp \overrightarrow{OP}$.

Translate $\overrightarrow{v}(t)$ to $\overrightarrow{OP'}$ at the origin: $\overrightarrow{OP'} = \overrightarrow{v}(t)$

$\overrightarrow{OP'} = (-y(t), x(t))$. Therefore,

$$\left(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t\right) = (x'(t), y'(t)) = \overrightarrow{v}(t) = \overrightarrow{OP'} = (-y(t), x(t)) = (-\sin t, \cos t)$$

So $\boxed{\frac{d}{dt}\cos t = -\sin t}$ and $\boxed{\frac{d}{dt}\sin t = \cos t}$

Derivatives of tangent and cotangent

Problem. Find the derivative of the functions $y = \tan x$ and $y = \cot x$.

Solution. $$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$
$$= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \frac{-(\tan x)'}{\tan^2 x} = \frac{-1/\cos^2 x}{\sin^2 x/\cos^2 x} = -\frac{1}{\sin^2 x}.$$

$$\boxed{\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}} \qquad \boxed{\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}}$$

Example 1

Example 1. Differentiate the functions
$$y = \sin(2x),\ y = \cos^2 x,\ y = \cos(x^2),\ y = \cos^2(x^2).$$

Solution. $$\frac{d}{dx}\sin(2x) = \cos(2x)\cdot\frac{d}{dx}(2x) = \cos(2x)\cdot(2) = 2\cos(2x).$$

$$\frac{d}{dx}\cos^2 x = 2\cos x\cdot\frac{d}{dx}(\cos x) = 2\cos x\cdot(-\sin x) = -2\cos x\sin x = -\sin(2x).$$

☞ use the double angle formula $\boxed{\sin(2x) = 2\sin x\cos x}$

$$\frac{d}{dx}\cos x^2 = -\sin x^2\cdot\frac{d}{dx}(x^2) = -\sin x^2\cdot(2x) = -2x\sin x^2.$$

$$\frac{d}{dx}\cos^2(x^2) = 2\cos(x^2)\cdot\frac{d}{dx}\cos(x^2)$$
$$= 2\cos(x^2)(-\sin(x^2))\cdot\frac{d}{dx}x^2 = 2\cos(x^2)(-\sin(x^2))(2x)$$
$$= -4x\cos(x^2)\sin(x^2) = -2x\sin(2x^2).$$

Example 2

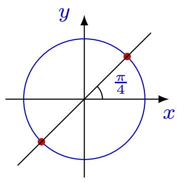
Example 2. Find all points where the graph of the function $f(x) = e^x \cos x$ has a horizontal tangent line.

Solution. The slope of a **horizontal** line is equal to 0.
Since the slope of the tangent line at a point x is equal to $f'(x)$,
we have to find the points x for which $f'(x) = 0$.

$$f'(x) = \frac{d}{dx}(e^x \cos x) = \frac{d}{dx}(e^x)\cos x + e^x \frac{d}{dx}(\cos x) = e^x \cos x - e^x \sin x$$
$$= (\cos x - \sin x)e^x .$$

Solve the equation $f'(x) = 0$:

$$(\cos x - \sin x)e^x = 0 \iff \cos x = \sin x \iff \tan x = 1 \iff x = \frac{\pi}{4} + \pi k, k \in \mathbb{Z} .$$



Answer: The graph has horizontal tangent lines

at points $x = \frac{\pi}{4} + \pi k, k \in \mathbb{Z}$.

High-order derivatives

Example 3. Find 2019th derivative of $y = \sin x$.

Solution.

$$\underbrace{\sin x \xrightarrow{\frac{d}{dx}} \cos x \xrightarrow{\frac{d}{dx}} -\sin x \xrightarrow{\frac{d}{dx}} -\cos x}_{\text{cycle of length } 4} \xrightarrow{\frac{d}{dx}} \sin x \xrightarrow{\frac{d}{dx}} \cdots$$

Divide 2019 by 4 with a remainder: $2019 = 4 \cdot 504 + 3$.

Therefore, $(\sin x)^{(2019)} = -\cos x$.

Example 4. Show that $y = \cos x$ is a solution of the differential equation $y'' + y = 0$.

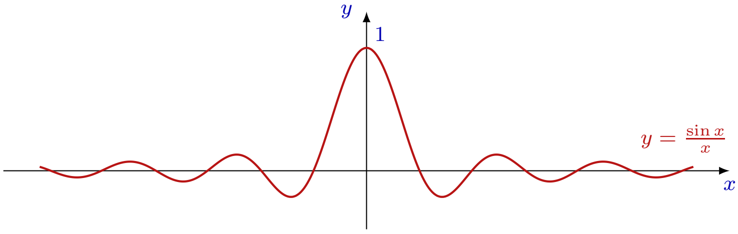
Solution. We have to show that $y = \cos x$ satisfies the equation $y'' + y = 0$.
$y' = (\cos x)' = -\sin x$, $y'' = (-\sin x)' = -\cos x$. Therefore,
$\underbrace{-\cos x}_{y''} + \underbrace{\cos x}_{y} = 0$, as required.

Using the derivative to calculate limits

Problem. Calculate the limit $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\sin(0+h) - \sin 0}{h} = \frac{d}{dx}(\sin x)\Big|_{x=0}$

$= \cos x\Big|_{x=0} = \cos 0 = 1\,.$



$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Summary

In this lecture, we learned the derivatives of trigonometric functions.
Here they are, for reference, and you should memorize them:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2 x} \\
\frac{d}{dx}\cot x &= -\frac{1}{\sin^2 x}
\end{aligned}$$

Comprehension checkpoint

- Find the derivatives of the following functions:

$y = \cos(2x^3 + 1)$, $y = \sin^3(2x)$, $y = \tan\dfrac{1}{x}$, $y = \cot(\ln x)$, $y = \sin(\cos x)$.

- Find all the points where the graph of the function $y = \dfrac{1}{\cos x}$ has a horizontal tangent line.

- Represent the limit $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x}$ as a **derivative** and calculate it.