Lecture 13

Differentiation Rules. Part 2

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Objectives

In this lecture we continue to develop tools for **efficient** computation of derivatives.
Namely, we

- calculate the derivative of exponential functions and
- establish the **chain rule** for differentiation of a composition of functions.

Also, we discuss the application of differentiation to differential equations.

The chain rule

The *chain rule* tells us how to differentiate a **composition** of functions.

Theorem (the chain rule).
If g is differentiable at x and f is differentiable at $g(x)$,
then $f \circ g$ is differentiable at x and $(f \circ g)'(x) = f'(g(x))g'(x)$.

In Leibniz notation: let $u = g(x)$ and $y = f(u)$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

More precisely, to indicate the point each derivative is evaluated at,
$\dfrac{dy}{dx}(x) = \dfrac{dy}{du}(u(x)) \cdot \dfrac{du}{dx}(x)$

If y depends on u
and u depends on x,
then y depends on x and
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$y = f(g(x)) \left(\begin{array}{ll} y & \\ \big| & y = f(u) \\ u & \\ \big| & u = g(x) \\ x & \end{array} \right.$$

Sketch of a proof

By the definition of the derivative,

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \underset{\substack{\uparrow \\ \text{if } \Delta u \neq 0}}{=} \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \frac{dy}{du}\frac{du}{dx}.$$

It may happen that $\Delta u = g(x+\Delta x) - g(x)$ vanishes.
Then the reasoning above is not valid, since we can't divide by 0.
But the proof may be adjusted to avoid this obstacle. See the textbook for the details.

Chain rule: Example 1

Example 1. Differentiate the function $f(x) = (x^4+3)^{10}$.

Solution. The function f is a **composition** of two functions: $g(x) = x^4+3$ and $f(u) = u^{10}$:

$$x \xrightarrow{g} x^4+3 \xrightarrow{f} (x^4+3)^{10}.$$

Let $u = g(x) = x^4+3$ and $y = f(g(x)) = f(u) = u^{10}$.

$$y = (x^4+3)^{10} \left(\begin{array}{l} y \\ \big|\, y = u^{10} \\ u \\ \big|\, u = x^4+3 \\ x \end{array} \right.$$

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{d(u^{10})}{du}\dfrac{d(x^4+3)}{dx} = 10u^9(4x^3)$

$$= 10(x^4+3)^9(4x^3) = 40x^3(x^4+3)^9.$$

Chain rule: Example 1

We have got that $\dfrac{d}{dx}(x^4+3)^{10} = 40x^3(x^4+3)^9$.

Remark. Usually, we don't use extra letter u while differentiating.

Let us write the chain rule in terms of **inner** and **outer** functions:

$$(\underbrace{f}_{\substack{\textbf{outer}\\ \text{function}}}(\underbrace{g}_{\substack{\textbf{inner}\\ \text{function}}}(x)))' = \underbrace{f'(g(x))}_{\substack{\text{derivative}\\ \text{of } \textbf{outer}\\ \text{function}}}\ \underbrace{g'(x)}_{\substack{\text{derivative}\\ \text{of } \textbf{inner}\\ \text{function}}}$$

The differentiation is written as follows:

$$\frac{d}{dx}(x^4+3)^{10} = \underbrace{10(x^4+3)^9}_{\substack{\text{derivative}\\ \text{of } \textbf{outer}\\ \text{function}}} \cdot \underbrace{(4x^3)}_{\substack{\text{derivative}\\ \text{of } \textbf{inner}\\ \text{function}}} = 40x^3(x^4+3)^9 .$$

Or

$$\frac{d}{dx}(x^4+3)^{10} = 10(x^4+3)^9 \cdot \frac{d}{dx}(x^4+3) = 10(x^4+3)^9(4x^3) = 40x^3(x^4+3)^9.$$

Chain rule: Example 2

Example 2. Find the derivative of $f(x) = \sqrt{x^2+1}$.

Solution. $f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$. Therefore,

$$\frac{d}{dx}(x^2+1)^{\frac{1}{2}} = \underbrace{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}} \cdot \underbrace{(2x)}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}} = \frac{x}{\sqrt{x^2+1}} .$$

Remark. One can write the differentiation as follows:

$$\frac{d}{dx}(x^2+1)^{\frac{1}{2}} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\frac{d}{dx}(x^2+1) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{\sqrt{x^2+1}}.$$

The chain rule for more than two functions

For a composition of three functions:

$\frac{d}{dx}f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$.

In Leibniz notation: if $y = h(x),\ z = g(y),\ w = f(z)$, then

$\frac{dw}{dx}(x) = \frac{dw}{dz}(z(y(x)))\frac{dz}{dy}(y(x))\frac{dy}{dx}(x)$, or, in short, $\frac{dw}{dx} = \frac{dw}{dz}\frac{dz}{dy}\frac{dy}{dx}$.

Similar formulas are valid for any number of composed functions.

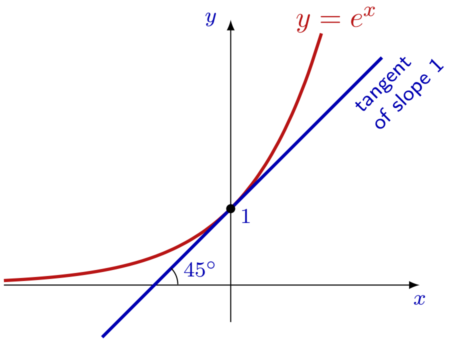
Example. Find the derivative of $f(x) = ((x^2+1)^3+2)^5$.

Solution.

$$\begin{aligned}\frac{d}{dx}((x^2+1)^3+2)^5 &= 5((x^2+1)^3+2)^4 \cdot \frac{d}{dx}((x^2+1)^3+2)\\ &= 5((x^2+1)^3+2)^4 \cdot 3(x^2+1)^2 \cdot \frac{d}{dx}(x^2+1)\\ &= 5((x^2+1)^3+2)^4 \cdot 3(x^2+1)^2 \cdot (2x) = 30x(x^2+1)^2((x^2+1)^3+2)^4.\end{aligned}$$

The derivative of e^x

The number e was defined as the base of the exponential function $f(x) = a^x$ whose graph has the tangent line at $(0,1)$ with slope 1.



The slope of this tangent line is the value of the derivative of $y = e^x$ at $x = 0$:

$y'(0) = \frac{d}{dx}e^x\Big|_{x=0} = 1$. How can we calculate the derivative of $y = e^x$ at any other point x?

The derivative of e^x

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = \lim_{h\to 0} e^x \cdot \lim_{h\to 0}\frac{e^h-1}{h}$$

$$= \lim_{h\to 0} e^x \cdot \lim_{h\to 0}\frac{e^{0+h}-e^0}{h} = e^x \cdot \underbrace{\frac{d}{dx}e^x\Big|_{x=0}}_{\substack{=1 \\ \text{by the definition of number } e}} = e^x .$$

$$\boxed{\frac{d}{dx}e^x = e^x}$$ The function is its own derivative!

Example 1. Find the derivative of $f(x) = e^{x^2+3x}$.

Solution. The function $f(x)$ is a composition of two functions, so we use the chain rule:

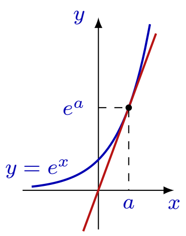
$$f'(x) = \frac{d}{dx}e^{x^2+3x} = e^{x^2+3x}\frac{d}{dx}(x^2+3x) = e^{x^2+3x}(2x+3).$$

The derivative of e^x

Example 2. Differentiate the function $f(x) = x^3e^{2x}$.
Solution. The function is a product of two functions, so we use the product rule.

$$f'(x) = \frac{d}{dx}(x^3e^{2x}) = \frac{d}{dx}(x^3)e^{2x} + x^3\frac{d}{dx}e^{2x} = 3x^2e^{2x} + x^3(e^{2x}\cdot 2) = (3x^2+2x^3)e^{2x}.$$

Example 3. Find the equation of the line that is tangent to graph of $y = e^x$ and passes through the origin.
Solution. We are looking for a line through the origin that is tangent to $y = e^x$.



The equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is $y - f(a) = f'(a)(x-a)$.
Since $f(a) = e^a$ and $f'(a) = \frac{d}{dx}e^x\Big|_{x=a} = e^x\Big|_{x=a} = e^a$,
the equation of the tangent is $y - e^a = e^a(x-a)$.
Since the tangent must passe through the origin,
the pair $(x,y) = (0,0)$ must satisfy the equation of the line:
$0 - e^a = e^a(0-a)$.
From which we get $-e^a = -ae^a$ or, equivalently, $(1-a)e^a = 0$. Since $e^a \neq 0$, this means $a = 1$.
Therefore, the equation of the tangent line is $y - e = e(x-1)$, or, equivalently, $y = ex$.

The derivative of a^x

Let us calculate the derivative of an arbitrary exponential function a^x, where $a > 0$.

By the properties of the logarithmic function, $a^x = e^{\ln(a^x)} = e^{x \ln a}$. Observe that $\ln a$ is a constant.

Differentiate a^x by the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\frac{d}{dx}(x\ln a) = e^{x\ln a}\cdot(\ln a) = a^x\ln a\,.$$

$$\boxed{\frac{d}{dx}a^x = a^x\ln a}$$

⚠ **Warning:** Don't mix up
the **power** function $y = x^a$ and the **exponential** function $y = a^x$.

They are very different functions and have different derivatives:

$$\frac{d}{dx}x^a = ax^{a-1}\,, \quad \frac{d}{dx}a^x = a^x\ln a\,.$$

Examples

Example 1. Find the derivative of $f(x) = x^3 + 3^x$.

Solution. $f'(x) = \frac{d}{dx}x^3 + \frac{d}{dx}3^x = 3x^2 + 3^x\ln 3\,.$

☞ Don't confuse power and exponential functions!

Example 2. Find the derivative of $f(x) = \dfrac{1}{2^{x^2-x}}$.

Solution. $f(x) = \dfrac{1}{2^{x^2-x}} = 2^{-x^2+x}$.

$$\begin{aligned} f'(x) = \frac{d}{dx}2^{-x^2+x} = 2^{-x^2+x}(\ln 2)\frac{d}{dx}(-x^2+x) &= 2^{-x^2+x}(\ln 2)(-2x+1) \\ &= (-2x+1)2^{-x^2+x}\ln 2\,. \end{aligned}$$

Example 3. Find the derivative of $f(x) = 2^{1/x}$

Solution. $$\begin{aligned} f'(x) = \frac{d}{dx}2^{1/x} = 2^{1/x}(\ln 2)\frac{d}{dx}\frac{1}{x} &= 2^{1/x}(\ln 2)(-\frac{1}{x^2}) \\ &= -\frac{2^{1/x}}{x^2}\ln 2. \end{aligned}$$

Differential equations

A *differential equation* (DE) is an equation involving derivatives of an unknown function.

A *solution* of a differential equation is a **function** satisfying the equation.

Example 1. $y' = 3x^2 - 1$ is a differential equation. It says that the derivative y' of an unknown function $y = y(x)$ is equal to $3x^2 - 1$.

The function $y = x^3 - x$ is a solution of this DE, since $y' = \frac{d}{dx}(x^3 - x) = 3x^2 - 1$.

Actually, this DE has infinitely many solutions:
any function $y(x) = x^3 - x + C$, where C is a constant, is a solution.

Indeed, $y'(x) = \frac{d}{dx}(x^3 - x + C) = 3x^2 - 1$.

In fact, these are **all** the solution of this DE.
The solution $y(x) = x^3 - x + C$, where C is an arbitrary constant,
is called the **general solution** of the DE.

Solutions of differential equations

Example 2. Show that the function $y = -x^3 + \frac{1}{x}$ is a solution of the differential equation $x^2y'' - xy' - 3y = 0$.

Solution. We have to show that the given function y and its derivatives y', y'' satisfy the differential equation. For this, find y' and y'':

$$y' = \frac{d}{dx}\left(-x^3 + \frac{1}{x}\right) = -3x^2 - \frac{1}{x^2}, \quad y'' = (y')' = \frac{d}{dx}\left(-3x^2 - \frac{1}{x^2}\right) = -6x + \frac{2}{x^3}.$$

Substitute y, y', y'' into the left hand side of the equation:

$$x^2y'' - xy' - 3y = x^2\underbrace{\left(-6x + \frac{2}{x^3}\right)}_{y''} - x\underbrace{\left(-3x^2 - \frac{1}{x^2}\right)}_{y'} - 3\underbrace{\left(-x^3 + \frac{1}{x}\right)}_{y}$$

$$= -6x^3 + \frac{2}{x} + 3x^3 + \frac{1}{x} + 3x^3 - \frac{3}{x} = 0 \ \checkmark$$

We see that the function $y = -x^3 + \frac{1}{x}$ **satisfies** the differential equation.
Therefore, it is a **solution** of this DE.

Summary

In this lecture, we learned

- how to differentiate a **composition** of several functions using the **chain rule**
- what the derivatives of the **exponential functions** are:

$\frac{d}{dx}a^x = a^x \ln a,$ in particular, $\frac{d}{dx}e^x = e^x$

- an application of differentiation: **differential equations** and their **solutions**.

Comprehension checkpoint

- Present the function $y = 2^{(3^x)}$ as a composition of two functions.

Find the derivative $\frac{d}{dx}2^{(3^x)}$.

- Present the function $y = ((5x^3 + 4)^2 + 1)^4$ as a composition of several functions. Find the derivative $\frac{d}{dx}((5x^3 + 4)^2 + 1)^4$.
- Find the derivatives $\frac{d}{dx}\sqrt{2}^x$ and $\frac{d}{dx}x^{\sqrt{2}}$.
- Show that the function $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$.