Lecture 12

Differentiation Rules. Part 1

Objectives

The derivative of a function $f(x)$ is defined as a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The direct application of this definition on an everyday basis would be tedious.

In this lecture we develop tools for the **efficient** computation of derivatives.

Namely, we

- calculate the derivative of power functions and
- establish differentiation rules that enable us to calculate the derivatives of more complicated functions, namely the sum, multiple, product, reciprocal and quotient rules.

Derivatives of power functions

In Lecture 11, we calculated the derivative of the linear function $f(x) = ax + b$, a, b are given constants: $\frac{d}{dx}(ax+b) = a$.

In particular, $\frac{d}{dx}C = 0$ for any constant C, and $\frac{d}{dx}x = 1$.

We also calculated the derivative of $f(x) = x^2$: $\frac{d}{dx}x^2 = 2x$.

Let us calculate the derivative of a **power** function $f(x) = x^n$ where n is a positive integer:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

use the formula: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

$$= \lim_{h \to 0} \frac{(x+h-x)((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1})}{h}$$

$$= \lim_{h \to 0} ((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1})$$

$$= x^{n-1} + x^{n-2}x + \cdots + xx^{n-2} + x^{n-1} = nx^{n-1}.$$

Derivatives of power functions

Therefore, $\frac{d}{dx}x^n = nx^{n-1}$ for any positive integer n.
We will prove later that this formula is valid for **any** real exponent:

$$\boxed{\frac{d}{dx}x^a = ax^{a-1} \text{ for any real } a} \quad \text{the power rule}$$

Example. Find the derivatives of the following functions:

$$x^{2021}, \ x^{\frac{3}{5}}, \ \sqrt{x}, \ x^{\sqrt{2}}, \ \sqrt[3]{x^7}, \ \frac{1}{x^4}.$$

Solution. All the functions above are **power** functions. We differentiate them according to the **power rule**:

$$\frac{d}{dx}x^{2021} = 2021x^{2020}, \qquad \frac{d}{dx}x^{\frac{3}{5}} = \tfrac{3}{5}x^{\frac{3}{5}-1} = \tfrac{3}{5}x^{-\frac{2}{5}},$$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}, \qquad \frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1},$$

$$\frac{d}{dx}\sqrt[3]{x^7} = \frac{d}{dx}x^{\frac{7}{3}} = \tfrac{7}{3}x^{\frac{7}{3}-1} = \tfrac{7}{3}x^{\frac{4}{3}} = \tfrac{7}{3}\sqrt[3]{x^4},$$

$$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}.$$

Differentiation rules

Let $f(x)$ and $g(x)$ be differentiable functions. Then

- Derivative of a sum: $(f+g)' = f' + g'$
- Derivative of a difference: $(f-g)' = f' - g'$
- Derivative of a product: $(fg)' = f'g + fg'$
- Derivative of a multiple: $(Cf)' = Cf'$
- Derivative of a reciprocal: $\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$
- Derivative of a quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

How to prove differentiation rules

All differentiation rules are deduced from the definition of the derivative.

Let us prove the rules. We start with the derivative of a constant multiple.

$$(Cf)'(x) = \lim_{h\to 0} \frac{(Cf)(x+h) - (Cf)(x)}{h} = \lim_{h\to 0} \frac{Cf(x+h) - Cf(x)}{h}$$

$$= C \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = Cf'(x)\,.$$

$$\boxed{(Cf)' = Cf'}$$

Example. Calculate the derivative of the function $f(x) = -7x^3$.

Solution. $f'(x) = (-7x^3)' = -7(x^3)' = -7(3x^2) = -21x^2$.

The derivative of a sum

The derivative of a sum is the sum of the derivatives. Indeed:

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)\,.$$

The derivative of a sum

The formula $(f+g)' = f' + g'$ extends easily to sums of any finite number of functions:

$(f_1 + f_2 + \cdots + f_n)' = f_1' + f_2' + \cdots + f_n'$.

A similar formula is valid for the difference.

Example. Calculate the derivative of the function $f(x) = 2x^3 - x + 4 + \dfrac{1}{x}$.

Solution. $f'(x) = \dfrac{d}{dx}\left(2x^3 - x + 4 + \dfrac{1}{x}\right) = \dfrac{d}{dx}(2x^3) - \dfrac{d}{dx}x + \dfrac{d}{dx}4 + \dfrac{d}{dx}(x^{-1})$

$$= 2(3x^2) - 1 + 0 + (-1)x^{-2} = 6x^2 - 1 - \frac{1}{x^2}\,.$$

The derivative of a product

$$(fg)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}\right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)\,.$$

$\boxed{(fg)' = f'g + fg'}$ the product rule

The product rule can be extended to any number of factors. For example,

$$(fgh)' = f'gh + fg'h + fgh'\,.$$

The derivative of a product: example

Example. Find the derivative of $f(x) = (x^2+1)(x-4)$.

Method 1. $f(x) = (x^2+1)(x-4) = x^3 - 4x^2 + x - 4$.

$$f'(x) = \frac{d}{dx}(x^3 - 4x^2 + x - 4) = 3x^2 - 8x + 1\,.$$

Method 2. $f'(x) = (\underbrace{(x^2+1)}_{f} \cdot \underbrace{(x-4)}_{g})'$

$$= \underbrace{\frac{d}{dx}(x^2+1)}_{f'} \cdot \underbrace{(x-4)}_{g} + \underbrace{(x^2+1)}_{f} \cdot \underbrace{\frac{d}{dx}(x-4)}_{g'}$$

$$= 2x(x-4) + (x^2+1)\cdot 1 = 2x^2 - 8x + x^2 + 1 = 3x^2 - 8x + 1\,.$$

☞ It's easier to differentiate a sum than a product!

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The derivative of a reciprocal

$$\left(\frac{1}{f(x)}\right)' = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x+h)f(x)}$$

$$= \lim_{h\to 0}\left(-\frac{f(x+h)-f(x)}{h} \cdot \frac{1}{f(x+h)f(x)}\right)$$

$$= -\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0}\frac{1}{f(x+h)f(x)}$$

$$= -f'(x)\cdot\frac{1}{f^2(x)} = \frac{-f'(x)}{f^2(x)}\,.$$

$\boxed{\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}}$ the reciprocal rule

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The derivative of a reciprocal: examples

Example 1. Differentiate the functions $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = \dfrac{1}{x}$.

Solution. $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x^2+1}\right) = \dfrac{-\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \dfrac{-2x}{(x^2+1)^2}$.

$g'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-\frac{d}{dx}x}{x^2} = -\dfrac{1}{x^2}$.

Example 2. Calculate the derivative of $f(x) = \dfrac{x-1}{x^2+x-2}$.

Solution. $f(x) = \dfrac{x-1}{x^2+x-2} = \dfrac{x-1}{(x-1)(x+2)} = \dfrac{\cancel{x-1}}{\cancel{(x-1)}(x+2)} = \dfrac{1}{x+2}$.

$f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x+2}\right) = \dfrac{-\frac{d}{dx}(x+2)}{(x+2)^2} = -\dfrac{1}{(x+2)^2}$.

☞ Simplify the expression before differentiation! whenever you can!

The derivative of a quotient

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)'$$

$$= f'(x)\frac{1}{g(x)} + f(x)\frac{-g'(x)}{g^2(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}}$$ the quotient rule

Example. Find the derivative of $f(x) = \dfrac{x+1}{2x^2-1}$.

Solution. $f'(x) = \dfrac{d}{dx}\left(\dfrac{x+1}{2x^2-1}\right) = \dfrac{\frac{d}{dx}(x+1)(2x^2-1) - (x+1)\frac{d}{dx}(2x^2-1)}{(2x^2-1)^2}$

$$= \frac{(2x^2-1)-(x+1)\cdot 4x}{(2x^2-1)^2} = \frac{-2x^2-4x-1}{(2x^2-1)^2}.$$

When the quotient rule is not the best choice

Example. Calculate the derivative of the function $f(x) = \dfrac{5x^2 - 3x + \sqrt{x}}{x^3}$.

Solution. We may use the quotient rule to differentiate this function,

but the calculation will be messy.

Instead, we convert the quotient into a sum and then differentiate the sum:

$$f(x) = \frac{5x^2 - 3x + \sqrt{x}}{x^3} = 5x^{-1} - 3x^{-2} + x^{-5/2}\,.$$

Now we are ready to differentiate:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^{-1} - 3x^{-2} + x^{-5/2}) \\ &= 5(-1)x^{-2} - 3(-2)x^{-3} + \left(-\frac{5}{2}\right)x^{-7/2} \\ &= -5x^{-2} + 6x^{-3} - \frac{5}{2}x^{-7/2}\,. \end{aligned}$$

☞ If the denominator of a quotient is a **monomial**,

break the quotient into a sum before differentiation.

Summary

In this lecture, we proved **important formulas** for differentiation, which you should memorize. They are listed here for reference.

$$\boxed{\begin{aligned} (f+g)' &= f' + g' \\ (f-g)' &= f' - g' \\ (fg)' &= f'g + fg' \\ (Cf)' &= Cf' \\ \left(\frac{1}{f}\right)' &= \frac{-f'}{f^2} \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \end{aligned}}$$

Remember also the **power rule**: $\boxed{\dfrac{d}{dx}x^a = ax^{a-1}}$

Comprehension checkpoint

- What is the fastest way to differentiate the function $f(x) = \dfrac{x^2 - 6x + 9}{x - 3}$?
- The same question for $f(x) = \dfrac{5}{x^3}$.
- Differentiate the function $f(x) = x(x+1)(x+2)$ in several different ways.

Make sure that you get the same result!

- Find the derivatives $\dfrac{d}{dx}\dfrac{\sqrt[3]{x}}{x-1}$ and $\dfrac{d}{dx}\dfrac{x-1}{\sqrt[3]{x}}$.