Lecture 11

The Derivative. Part 2

Objectives

In this lecture, we continue to study the **derivative** of a function.

We discuss

- the derivative as a function,
- calculating derivatives from the definition,
- Leibniz notation for the derivative,
- higher-order derivatives,
- differentiability and continuity.

The derivative as a function

The derivative of a function $f(x)$ at the point $x = a$ is a **number**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} .$$

If we let the point a vary and call it x, then we get the derivative at **any** point x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(where the limit exists).}$$

Given a number x, this gives a number $f'(x)$.

Therefore, we may regard f' as a new **function** of x: $x \mapsto f'(x)$.

This function is called the derivative of f.

Definition. The *derivative* of a function f is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' consists of all points x for which the limit exists (as a finite real number).

Velocity and acceleration as derivatives

Example 1. Take a point moving along a straight line, and let $s(t)$ be the position of the point on the line at the time $t.$

Then the derivative $s'(t)$,
which is the function giving the instantaneous rate of change of $s(t),$ Iis the **velocity** function $v(t)$:

$$v(t) = s'(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}.$$

The derivative $v'(t)$,
which is the function giving the instantaneous rate of change of $v(t),$
is the **acceleration** function $a(t)$:

$$a(t) = v'(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

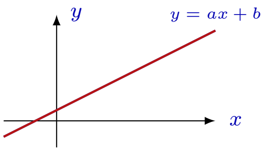
The derivative of a linear function

Example 2. Show that a linear function $f(x) = ax + b$,
where a, b are given constants, is differentiable for all x . Find $f'(x)$.

Solution. Differentiability is the existence of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(a(x+h)+b) - (ax+b)}{h} = \lim_{h \to 0} \frac{ah}{h} = \lim_{h \to 0} a = a\,.$$

Answer: The function $f(x) = ax + b$ is differentiable for all x .
Its derivative is $f'(x) = a$ (a constant function).



The tangent line coincides with the original line $y = ax + b$.

Remark. A constant function $y = C$ is a linear function $y = ax + b$ with $a = 0$ and $b = C$.
So its derivative is zero.

The derivative of $f(x) = x^2$

Example 3. Show that $f(x) = x^2$ is differentiable for all x. Find f'.

Solution. $f(x) = x^2$ is differentiable if the difference quotient has a limit:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - (x^2)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

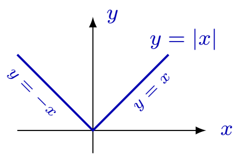
$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x.$$

Answer: Yes, $f(x) = x^2$ is differentiable for all x and $f'(x) = 2x$.

The derivative of $f(x) = |x|$

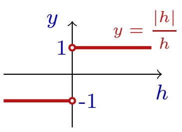
Example 4. At which points is $f(x) = |x|$ differentiable? Find f'.

Solution. Look at the graph of $y = |x|$:



The tangent line to the graph at $x = a > 0$ is $y = x$.
The tangent line to the graph at $x = a < 0$ is $y = -x$.
It's unclear what is the tangent line to $y = |x|$ at $x = 0$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$ This limit does not exist:



Therefore, $f(x) = |x|$ is **not** differentiable at $x = 0$ and the tangent line at $x = 0$ to the graph of $y = |x|$ does not exist.

Overall, for $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ we have $f'(x) = \begin{cases} 1, \text{ if } & x > 0 \\ \text{undefined, if } & x = 0 \\ -1, \text{ if } & x < 0. \end{cases}$

Leibniz notation for the derivative

Besides f', there other notations for the derivative of a function f.
Let $\Delta x = (x+h) - x = h$ be the change of x,
and $\Delta y = f(x+h) - f(x)$ be the corresponding change of y.

Then $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

Leibniz proposed to use the notation $\dfrac{dy}{dx}$ for the derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Leibniz notation proved its relevance over time and is used all through calculus. It helps to see the mathematical essence behind the symbols.
Get used to various notations for the derivative of a function $y = f(x)$:

$y', \; f'(x), \; \dfrac{dy}{dx}, \; \dfrac{d}{dx}(y), \; \dfrac{df}{dx}, \; \dfrac{d}{dx}f(x)$.

Leibniz notation

The derivative $\dfrac{dy}{dx}$ is the result of differentiation of the function $y = y(x)$.

If we denote the **operation** of taking the derivative (differentiation) by the symbol $\dfrac{d}{dx}$, then the derivative $\dfrac{dy}{dx}$ is the result of applying the operation $\dfrac{d}{dx}$ to the function y: $\dfrac{dy}{dx} = \dfrac{d}{dx}y$.

Differentiation is the process of taking the derivative: $y \xrightarrow{\frac{d}{dx}} \dfrac{dy}{dx}$.

Example. We know that $\dfrac{d}{dx}(x^2) = 2x$, $\dfrac{d}{dx}(ax+b) = a$, and $\dfrac{d}{dx}(C) = 0$.

In particular, $\dfrac{d}{dx}(2x) = 2$ and $\dfrac{d}{dx}(2) = 0$.

This is an example of **consecutive** differentiation:

$$x^2 \xrightarrow{\frac{d}{dx}} 2x \xrightarrow{\frac{d}{dx}} 2 \xrightarrow{\frac{d}{dx}} 0.$$

Higher-order derivatives

If the derivative f' of a function f is itself differentiable,
then its derivative $(f')'$ is called the *second derivative* of f and denoted by f''.

The derivative of the second derivative is called the third derivative, and so on.

$$f \xrightarrow{\frac{d}{dx}} f' \xrightarrow{\frac{d}{dx}} f'' \xrightarrow{\frac{d}{dx}} f''' \xrightarrow{\frac{d}{dx}} f^{(4)} \xrightarrow{\frac{d}{dx}} f^{(5)} \xrightarrow{\frac{d}{dx}} \dots \xrightarrow{\frac{d}{dx}} f^{(n)} \xrightarrow{\frac{d}{dx}}$$

In Leibniz notation: if $y = f(x)$, then $y' = \dfrac{dy}{dx} = f'(x)$,

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)\,,\ y''' = \frac{d^3y}{dx^3} = f'''(x)\,,\ y^{(4)} = \frac{d^4y}{dx^4} = f^{(4)}(x),\ \dots$$

The derivative of order zero is the function itself: $f^{(0)} = f$.

Example. As we have already seen, if $f(x) = x^2$, then

$$f' = 2x\,,\ f'' = (f')' = (2x)' = 2\,,\ f''' = (f'')' = (2)' = 0\,,\ f^{(4)} = f^{(5)} = \dots = 0$$

$$\underbrace{x^2}_{f} \xrightarrow{\frac{d}{dx}} \underbrace{2x}_{f'} \xrightarrow{\frac{d}{dx}} \underbrace{2}_{f''} \xrightarrow{\frac{d}{dx}} \underbrace{0}_{f'''} \xrightarrow{\frac{d}{dx}} \underbrace{0}_{f^{(4)}}$$

Velocity, acceleration, and jerk

If an object moves along a straight line,
and the function $s(t)$ describes the position of the object at time t,
then the derivative of the position function is the **velocity**: $s'(t) = v(t)$.

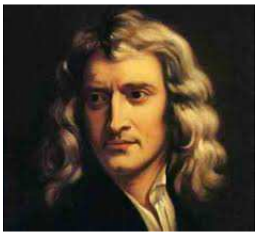
This means that velocity is the rate of change of position.

The rate of change of the velocity is called the *acceleration*: $a(t) = v'(t) = s''(t)$

The rate of change of the acceleration is called the *jerk*: $j(t) = a'(t) = v''(t) = s'''(t)$

$$s(t) \xrightarrow{\frac{d}{dt}} v(t) \xrightarrow{\frac{d}{dt}} a(t) \xrightarrow{\frac{d}{dt}} j(t)$$

Newton came to the definition of derivative
by studying how velocity varies with position.
He used the dot notation for the derivative:
$v = \dot{s}\,,\ a = \dot{v} = \ddot{s}\,,\ j = \dot{a} = \ddot{v} = \dddot{s}$



The derivative in the natural sciences

As we know, the derivative $\frac{dy}{dx} = f'(x)$ of a function $y = f(x)$ is the (instantaneous) rate of change of f.

Examples of rates of change.

1. If s is position at time t, then $\frac{ds}{dt} = v(t)$ is velocity and $\frac{dv}{dt} = a(t)$ is acceleration.

2. If $m(x)$ is the mass of a straight rod between the points 0 and x, then $\frac{dm}{dx} = \rho(x)$ is the (linear) density.

3. If Q is the electric charge in a capacitor at time t, then $\frac{dQ}{dt} = I(t)$ is the current flowing into the capacitor.

4. If P is the size of a population at time t, then $\frac{dP}{dt}$ is the rate of population growth/decrease.

Examples of rates of change

5. If N is the number people infected by a disease at time t, then $\frac{dN}{dt}$ is the rate of the spread of the disease.

6. If C is the total cost of x units of a product, then $\frac{dC}{dx} = m(x)$ is the marginal cost.

Differentiability and continuity

Remember that

- A function $f(x)$ is **continuous** at a point $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$.
- A function $f(x)$ is **differentiable** at a point $x = a$ if the following limit exists:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (= f'(a)).$$

Continuity and differentiability are important properties of a function.

How are they related to each other?

Differentiability implies continuity

Theorem. Differentiability implies continuity, that is
if a function is differentiable at a point, then it is continuous at this point.

Proof. We have to prove that if a function f is differentiable at a point $x = a$,
then it is continuous at $x = a$.

If a function f is differentiable at a point $x = a$, then $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a)$ exists.

We have to prove that $\lim\limits_{x \to a} f(x) = f(a)$. Consider the difference $\left(\lim\limits_{x \to a} f(x)\right) - f(a)$:

$$\left(\lim_{x \to a} f(x)\right) - f(a) = \lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) =$$

Let $h = x - a$, then $x = a + h$ and $x \to a \iff h \to 0$

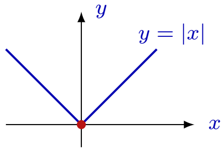
$$= \lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h} \cdot h\right) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h = f'(a) \cdot 0 = 0.$$

Therefore, $\lim\limits_{x \to a} f(x) - f(a) = 0$, which means that $\lim\limits_{x \to a} f(x) = f(a)$, as required.

Continuity does not imply differentiability

We have just proven that differentiability implies continuity.
The converse is **not** true: continuity does **not** imply differentiability.

Example 1. Consider $f(x) = |x|$.



As we already seen (p. 7), $f'(0)$ doesn't exist.
So the function is not differentiable at $x = 0$.
But it is continuous for all x, in particular, for $x = 0$.

Example 2. Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Is f continuous at $x = 0$?
Is f differentiable at $x = 0$?

Solution. For continuity, we have to check that $\lim_{x \to 0} f(x) = f(0)$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = ?$$

Example 2 (cont.): a piecewise defined function

For any $x \neq 0$, $-1 \leq \sin \dfrac{1}{x} \leq 1$. Therefore, $-x^2 \leq x^2 \sin \dfrac{1}{x} \leq x^2$

Since $-x^2 \underset{x \to 0}{\longrightarrow} 0$ and $x^2 \underset{x \to 0}{\longrightarrow} 0$, the Squeeze theorem gives us $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0$.

Therefore, $\lim_{x \to 0} f(x) = 0 = f(0)$, and f is **continuous** at 0.

Now check differentiability:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = ?$$

We will use the Squeeze theorem again: for any $h \neq 0$, $-1 \leq \sin \dfrac{1}{h} \leq 1$. Therefore, $-|h| \leq h \sin \dfrac{1}{h} \leq |h|$.
Since $-|h| \underset{h \to 0}{\longrightarrow} 0$ and $|h| \underset{h \to 0}{\longrightarrow} 0$,

we get $\lim_{h \to 0} h \sin \dfrac{1}{h} = 0$. Therefore, $f'(0)$ exists and f is differentiable at 0.

Remark. If we investigated differentiability first and obtained a positive answer, then checking continuity would be superfluous, since differentiability implies continuity.

Not continuous $\implies$ not differrentiable

As we know, if a function is differentiable, then it is continuous:

$$f \text{ is differentiable } \implies f \text{ is continuous.}$$

Logically that is equivalent to

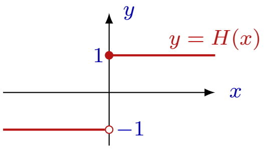
$$f \text{ is not differentiable } \impliedby f \text{ is not continuous.}$$

That is, if a function is not continuous, then it's not differentiable.

Differentiability is **stronger** than continuity.

Example. Is the Heaviside function $H(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ differentiable at 0?

Solution.



The function $H(x)$ is **not** continuous at $x = 0$, since $\lim\limits_{x \to 0^+} H(x) = 1$ and $\lim\limits_{x \to 0^-} H(x) = -1$.

Therefore, $H(x)$ is **not** differentiable at 0.

Summary

In this lecture we studied

- the derivative as a function
- the derivatives of the simplest functions:

$$\frac{d}{dx}(ax+b) = a, \quad \frac{d}{dx}C = 0, \quad \frac{d}{dx}x^2 = 2x, \quad \frac{d}{dx}(|x|) = \begin{cases} 1, \text{ if } & x > 0 \\ \text{undefined, if } & x = 0 \\ -1, \text{ if } & x < 0. \end{cases}$$

- Leibniz notation $\dfrac{dy}{dx}$ for the derivarive
- higher-order derivatives
- examples of rates of change in the natural sciences
- relationship between differentiability and continuity:

$$f \text{ is differentiable } \implies f \text{ is continuous}$$

$$f \text{ is differentiable } \not\Longleftarrow f \text{ is continuous.}$$

Comprehension checkpoint

- Find the following derivatives:

$$\frac{d}{dx}(-x+2), \quad \frac{d}{dx}(5x), \quad \frac{d}{dx}(1), \quad \frac{d}{dx}\left(\frac{2x+4}{5}\right), \quad \frac{d}{dx}x^2, \quad \frac{d}{dx}x.$$

- At which points is the function $f(x) = |x+3|$ not differentiable?
- What is $f'(-3)$ if $f(x) = x^2$?
- Find the second derivative of $f(x) = x^2$.
- What is $f^{(4)}$ if $f(x) = C$?
- Is it true that any continuous function is differentiable?
- Is it true that any differentiable function is continuous?