Lecture 10

Derivative. Part 1

Objectives

The notion of derivative of a function is one of the **central** concepts of calculus.

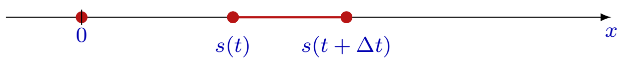
No derivatives, no calculus!

Calculus was born when the notion of derivative was formed.
This happened in the 17th century as the result of the work
of the founding fathers of calculus, Isaac Newton and Gottfried Wilhelm Leibniz.
We will study derivatives according to the following **plan**:

- Understanding the derivative (the topic of this lecture)
  - Kinematic interpretation (velocity)
  - Derivative as rate of change
  - Definition
  - Geometric interpretation (tangent line)
- Calculating of derivatives
  - Differentiation rules
  - Derivatives of elementary functions
- Applications of the derivative

Instantaneous velocity

Consider a point moving along a straight line:



Let $s(t)$ be the position of the point on the line at time t.

What is the **average velocity** of the point over the time interval $[t, t+\Delta t]$?

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{\Delta s}{\Delta t}$$

What happens when $\Delta t \to 0$?

The limit of average velocity is the **instantaneous velocity**:

$$v(t) = \lim_{\Delta t \to 0} v_{\text{ave}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

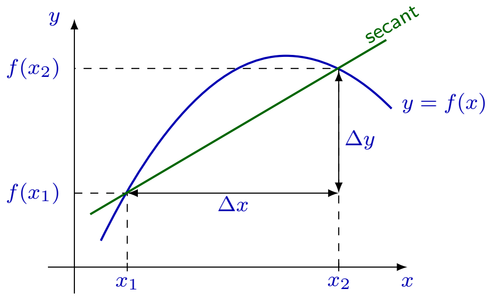
The quotient $\dfrac{\Delta s}{\Delta t} = \dfrac{s(t+\Delta t) - s(t)}{\Delta t}$ is called the **rate of change** of the function $s(t)$ on the interval $[t, t+\Delta t]$.

Average rate of change

Let $y = f(x)$ be a function. If x changes from x_1 to x_2, then the change in x (called the variable **increment**) is $\Delta x = x_2 - x_1$.

This change causes the change in y (the function **increment**): $\Delta y = f(x_2) - f(x_1)$.

The quotient $\dfrac{\Delta y}{\Delta x}$ is called the **difference quotient** or the **average rate of change** of y over the interval $[x_1, x_2]$.



Geometrically, the difference quotient $\dfrac{\Delta y}{\Delta x}$ is the **slope** of the secant line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Instantaneous rate of change

The limit of the average rate of change of a function is called the **instantaneous rate of change** (or simply the **rate of change**):

rate of change $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

For example, if an object moves along a straight line and $s(t)$ is the position function, pause then the instantaneous rate of change of s is the (instantaneous) velocity:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

The instantaneous rate of change of the velocity $v(t)$ is called the acceleration:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}.$$

The instantaneous rate of change of a function is a very important characteristic, it is called the **derivative** of the function.

Definition of the derivative

Definition. A function $f(x)$ is said to be *differentiable* at a point $x = a$

if the limit $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

In this case the number

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is called the *derivative* of $f(x)$ at $x = a$ and denoted by $f'(a)$.

Using Δ-notations, we may write:

$\Delta a = (a+h) - a = h$ and $\Delta f(a) = f(a+h) - f(a)$. Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{\Delta a \to 0} \frac{\Delta f(a)}{\Delta a}.$$

The **derivative** of a function at a point is the **rate of change** of the function at this point.

Calculating the derivative from the definition

Example 1. Find the rate of change of the function $f(x) = x^2$ at the point $x = 1$.

Solution. The rate of change of a function at a point is the derivative of the function at this point. Therefore, we have to calculate $f'(1)$ for $f(x) = x^2$ using the **definition** of the derivative:

$f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ for $f(x) = x^2$ and $a = 1$. Do the math:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}$$
$$= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} = \lim_{h \to 0} (2+h) = 2.$$

We have got that $f'(1) = 2$.

Therefore, the rate of change of $f(x) = x^2$ at the point $x = 1$ is $\boxed{2}$.

When the rate of change is zero, or a constant

Example 2. Find the derivative of $f(x) = C,$ where C is a constant, at an arbitrary point $x = a.$

Solution. Since $f(a+h) = f(a) = C,$ we get

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{C - C}{h} = \lim_{h\to 0} \frac{0}{h} = \boxed{0}$$

This result means that the rate of change of a constant function at an arbitrary point is **zero**.
This makes sense, since the function doesn't change!

Example 3. Show that the function $f(x) = x$ has the same rate of change at all points.

Solution. Calculate the derivative of the function at an arbitrary point $x = a:$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h) - a}{h} = \lim_{h\to 0} \frac{h}{h} = \boxed{1}$$

We see that the rate of change equals 1 independently of the point.

The derivative of a radical function

Example 4. Find the derivative of $f(x) = \sqrt{x}$ at $x = 3$.

Solution. $f'(3) = \lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{3+h} - \sqrt{3}}{h}$

[We need to rationalize the quotient]

$$= \lim_{h\to 0} \frac{(\sqrt{3+h} - \sqrt{3})(\sqrt{3+h} + \sqrt{3})}{h(\sqrt{3+h} + \sqrt{3})} = \lim_{h\to 0} \frac{(\sqrt{3+h})^2 - (\sqrt{3})^2}{h(\sqrt{3+h} + \sqrt{3})}$$

$$= \lim_{h\to 0} \frac{3 + h - 3}{h(\sqrt{3+h} + \sqrt{3})} = \lim_{h\to 0} \frac{1}{\sqrt{3+h} + \sqrt{3}} = \frac{1}{\sqrt{3} + \sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$$

When is a function not differentiable

Example 5. Show that the function $f(x) = |x|$ is **not** differentiable at the point $x = 0$.

Solution. We have to show that $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ does not exist.

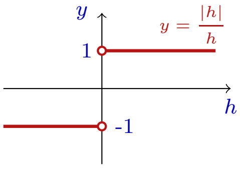
For $f(x) = |x|$ and $a = 0$ we have

$\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \dfrac{|0+h|-|0|}{h} = \lim_{h\to 0} \dfrac{|h|}{h}$. This limit does not exist,

since the right- and the left-hand limits exist but do not coincide:

$\lim_{h\to 0^+} \dfrac{|h|}{h} = \lim_{h\to 0^+} \dfrac{h}{h} = \lim_{h\to 0^+} 1 = 1$ and

$\lim_{h\to 0^-} \dfrac{|h|}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$.



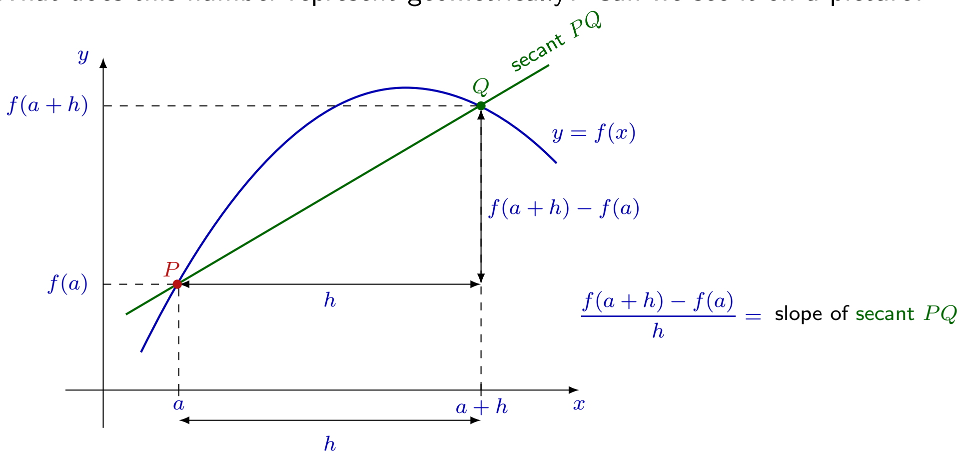
Therefore, $f(x) = |x|$ is **not** differentiable at $x = 0$.

The geometric meaning of the derivative

By definition, the derivative of a function $f(x)$ at a point $x = a$ is the number

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

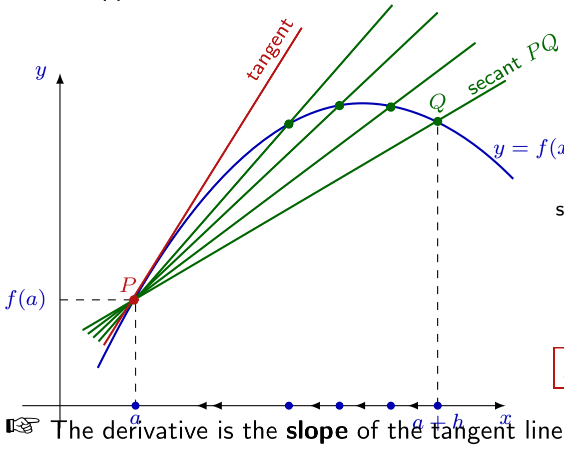
What does this number represent geometrically? Can we see it on a picture?



The geometric meaning of the derivative

We have found that $\dfrac{f(a+h)-f(a)}{h}$ is
the slope of the secant line through $P(a, f(a))$ and $Q(a+h, f(a+h))$.

What happens when $h \to 0$?



secant line $\xrightarrow[h\to 0]{}$ tangent line

slope of secant line $\xrightarrow[h\to 0]{}$ slope of tangent line

$$\begin{array}{ccc} \| & & \| \\ \dfrac{f(a+h)-f(a)}{h} & \xrightarrow[h\to 0]{} & f'(a) \end{array}$$

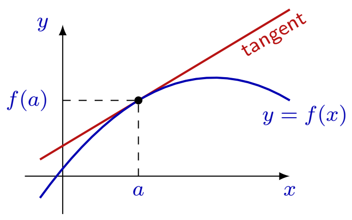
$\boxed{f'(a) = \text{the slope of the tangent line}}$

☞ The derivative is the **slope** of the tangent line.

The equation of the tangent line

We saw that the **derivative** $f'(a)$ of a function $f(x)$ at a point $x = a$
is equal to the **slope** of the tangent line to the graph of the function at the point $(a, f(a))$.

So if a function is differentiable at a point,
then there exists the tangent line to the graph of the function at this point.



What is the equation of this tangent line?
We know that the equation of the line that passes through the point $(a, f(a))$ and has slope m
is $y - f(a) = m(x-a)$.
We know also that the slope $m = f'(a)$.
So $y - f(a) = f'(a)(x-a)$.

☞ The equation of the tangent line to the graph of the function $y = f(x)$
at the point $(a, f(a))$ is $\boxed{y - f(a) = f'(a)(x-a)}$

The equation of the tangent line: example

Example. Find the equation of the tangent line to the graph of $y = x^2$ at the point $x = 1$.

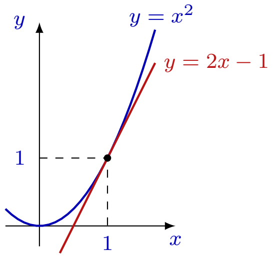
Solution. The equation of the tangent line is $y - f(a) = f'(a)(x - a)$.
We are given: $f(x) = x^2$ and $a = 1$.
To write the equation, we need to know numbers $f(a)$ and $f'(a)$.

$f(a) = f(1) = 1^2 = 1$, $f'(a) = f'(1) = ?$

$f'(1)$ has already been calculated (see page 7): $f'(1) = 2$.

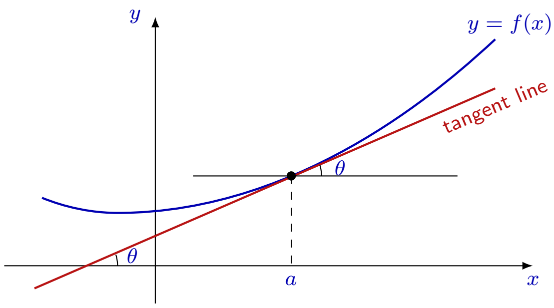
Putting $a = 1$, $f(a) = 1$, $f'(a) = 2$
into the equation of the tangent line, we get
$y - 1 = 2(x - 1)$, or, equivalently, $y = 2x - 1$.

Therefore, the equation of the tangent line
to $y = x^2$ at $x = 1$ is $\boxed{y = 2x - 1}$



Derivative, slope and angle of incline

Let $f(x)$ be a function differentiable at the point $x = a$.
This means that its graph has a tangent line at $x = a$.



The slope m of the tangent line is equal to $\tan\theta$,
where θ is the angle of incline of the tangent line: $m = \tan\theta$.
The slope m of the tangent line is equal to the derivative of f at a: $m = f'(a)$.
Therefore, $f'(a) = m = \tan\theta$. Note that $\theta \neq 90°$.

The angle of incline: example

Problem. Find a point on the graph of the hyperbola $f(x) = \frac{1}{x}$ where the tangent line makes an angle of $135°$ with the positive direction on the x-axis.

Solution. The angle of incline is $\theta = 135°$.

This means that we are looking for a point $(x, f(x))$ on the hyperbola where $f'(x) = \tan 135° = -1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{(x+h)xh} = \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = -\frac{1}{x^2}.$$

We need to find the value of x for which $f'(x) = -1$, that is $-\dfrac{1}{x^2} = -1$.

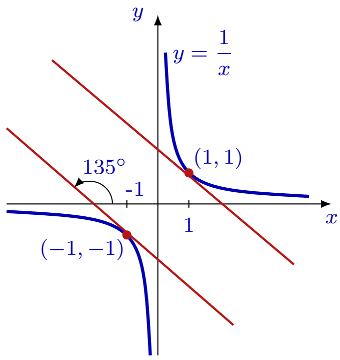
In fact, there are two:

$-\dfrac{1}{x^2} = -1 \iff \dfrac{1}{x^2} = 1 \iff x = 1$ or $x = -1$.

The angle of incline: example

Therefore, there are **two** points on the hyperbola $y = \dfrac{1}{x}$ where the tangent line has the slope -1:

$(x, y) = (1, 1)$ and $(x, y) = (-1, -1)$:



Summary

In this lecture we learned

- the definition of the **derivative** of a function $f(x)$ at a point $x = a$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

- the geometric interpretation of the derivative as the **slope** of the tangent line
- the equation of the **tangent line** to the graph of the function $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

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Comprehension checkpoint

- Why is the derivative of a constant function equal to zero at any point?
- Do you remember the definition of the derivative of a function at a point? Write it down!
- What is the equation of the tangent line to the graph of $y = x$ at the point $x = 1$?
- The graph of the function $y = f(x)$ has a horizontal tangent line at the point $x = 2$. Find $f'(2)$.

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