Lecture 9

Limits at Infinity

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Objectives

There are two types of limits involving infinity:

- infinite limits $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$ and
- limits at infinity $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = L$.

In this lecture we discuss

- **limits at infinity**,
- calculation **techniques** for limits at infinity,
- applications for finding horizontal and oblique **asymptotes** for the graphs of functions.

Limits at infinity

Definition. Let $f(x)$ be a function and L be a number.
One says that f has *limit* L *as* x *approaches infinity*
if $f(x)$ is arbitrary close to L whenever x is a sufficiently large number.

Notation: $\lim_{x\to\infty} f(x) = L$ or $f(x) \underset{x\to\infty}{\longrightarrow} L$.

Similarly, we say that f has *limit* L *as* x *approaches negative infinity*
if $f(x)$ is arbitrary close to L whenever x is a sufficiently large negative number.

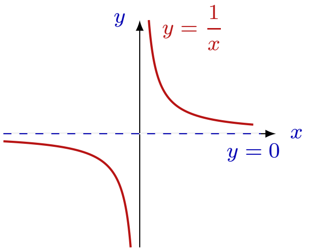
Notation: $\lim_{x\to-\infty} f(x) = L$ or $f(x) \underset{x\to-\infty}{\longrightarrow} L$.

⚠ **Warning:** ∞ and $-\infty$ are **not** real numbers, and can not be treated as real numbers.

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$
is called a *horizontal asymptote* for the graph of $y = f(x)$.

Standard examples of limits at infinity

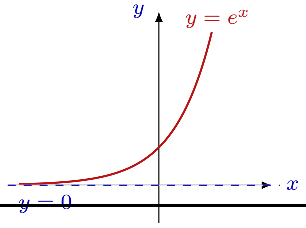
Example 1.



$$\lim_{x\to\infty} \frac{1}{x} = 0\,, \quad \lim_{x\to-\infty} \frac{1}{x} = 0\,.$$

$y = 0$ is a **horizontal asymptote** for $y = \dfrac{1}{x}$

Example 2.



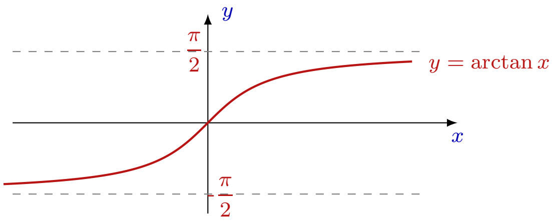
$$\lim_{x\to-\infty} e^x = 0\,,$$

$y = 0$ is a **horizontal asymptote** for $y = e^x$

Remember: $e^x \neq 0$.

Standard examples of limits at infinity

Example 3.



$\displaystyle\lim_{x\to\infty} \arctan x = \frac{\pi}{2}\,,$ so $y = \dfrac{\pi}{2}$ is a **horizontal asymptote** for $y = \arctan x$.

$\displaystyle\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}\,,$ so $y = -\dfrac{\pi}{2}$ is a **horizontal asymptote** for $y = \arctan x$.

Calculating techniques for limits involving infinity

There are various techniques for calculating limits involving infinity.

These techniques depend on the functions whose limits we need to find.

Below we will discuss how to calculate

- limits of the **standard** functions, like e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\arctan x$, etc,
- limits of **rational functions**: those of type $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials,
- limits involving **irrational expressions**, like radicals $\sqrt{x+1}-\sqrt{x}$,
- limits by the **squeeze theorem**.

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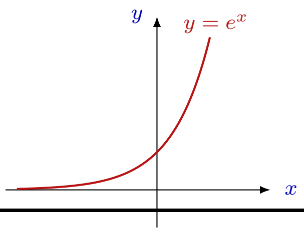
Limits of standard functions

Success in calculating these types of limits depends on your knowledge of the basic elementary functions.

Example 1. Calculate $\lim\limits_{x\to\infty} e^x$, $\lim\limits_{x\to-\infty} e^x$, $\lim\limits_{x\to\infty} e^{-x}$, $\lim\limits_{x\to-\infty} e^{-x}$.

Solution. Draw and see:



$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0,$$

$$\lim_{x\to\infty} e^{-x} = \lim_{t\to-\infty} e^t = 0,$$

$$\lim_{x\to-\infty} e^{-x} = \lim_{t\to\infty} e^t = \infty.$$

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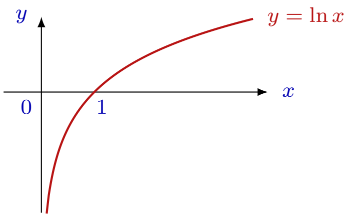
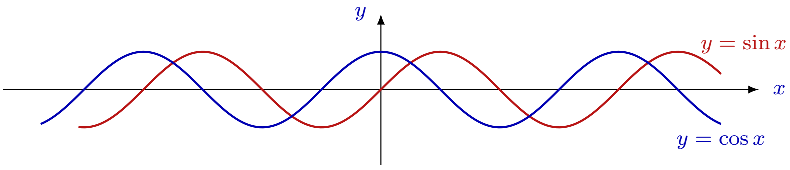
Draw and see

Example 2. Calculate $\lim\limits_{x\to\infty} \ln x$, $\lim\limits_{x\to\infty} \sin x$, $\lim\limits_{x\to-\infty} \cos x$.

Solution. Draw and see:



$$\lim_{x\to\infty} \ln x = \infty$$



$\lim\limits_{x\to\infty} \sin x$ does not exist, $\lim\limits_{x\to-\infty} \cos x$ does not exist.

Limits of compositions

Example. Evaluate $\lim\limits_{x\to\infty} e^{\frac{1}{x}}$ and $\lim\limits_{x\to\infty} \ln \frac{1}{x^2}$

Solution.

$$\lim_{x\to\infty} e^{\frac{1}{x}} \begin{bmatrix} t = \dfrac{1}{x} \\ x \to \infty \implies t \to 0 \end{bmatrix} = \lim_{t\to 0} e^t = e^0 = 1.$$

$$\lim_{x\to\infty} \ln \frac{1}{x^2} \begin{bmatrix} t = \dfrac{1}{x^2} \\ x \to \infty \implies t \to 0 \end{bmatrix} = \lim_{t\to 0^+} \ln t = -\infty.$$

There is another way to calculate the latter limit:

Since $\ln \dfrac{1}{x^2} = \ln x^{-2} = -2\ln x$ for $x > 0$, we get

$$\lim_{x\to\infty} \ln \frac{1}{x^2} = \lim_{x\to\infty} (-2\ln x) = -2 \lim_{x\to\infty} \ln x = -\infty.$$

Limits at infinity of rational functions

Example 1 (numerator and denominator of the same degree).

Evaluate $\lim_{x\to\infty} \dfrac{2x^3 - x^2 + 5x}{4x^3 - x - 1}$.

Solution. Both numerator and denominator go to infinity when $x \to \infty$.

Which one goes to infinity faster?

The outcome of the race is determined by the highest power of x.

Here is a trick that will help you to find limits at infinity of rational functions:

☞ Divide both numerator and denominator

by the highest power of x appearing in the denominator:

$$\lim_{x\to\infty} \frac{2x^3 - x^2 + 5x}{4x^3 - x - 1} \left[\text{divide by } x^3 \right]$$

$$= \lim_{x\to\infty} \frac{2 - \frac{1}{x} + 5\frac{1}{x^2}}{4 - \frac{1}{x^2} - \frac{1}{x^3}} = \lim_{x\to\infty} \frac{2 - 0 + 0}{4 - 0 - 0} = \frac{1}{2}.$$

Limits at infinity of rational functions

Example 2 (degree of numerator less than degree of denominator).

Evaluate $\lim_{x\to-\infty} \dfrac{2x + 1}{x^3 + 2x^2 - 1}$.

Solution. Both numerator and denominator go to negative infinity when $x \to -\infty$.
The denominator gets there faster!

☞ Divide both numerator and denominator

by the highest power of x appearing in the denominator:

$$\lim_{x\to-\infty} \frac{2x + 1}{x^3 + 2x^2 - 1} \left[\text{divide by } x^3 \right]$$

$$= \lim_{x\to-\infty} \frac{2\frac{1}{x^2} + \frac{1}{x^3}}{1 + 2\frac{1}{x} - \frac{1}{x^3}} = \lim_{x\to-\infty} \frac{0 + 0}{1 + 0 - 0} = \frac{0}{1} = 0.$$

Limits at infinity of rational functions

Example 3 (degree of numerator greater than degree of denominator).

Evaluate $\lim\limits_{x \to -\infty} \dfrac{x^2+1}{x+2}$.

Solution. When $x \to -\infty$, the numerator goes to infinity and the denominator goes to negative infinity. Where does the fraction go?

☞ Divide both numerator and denominator by the highest power of x appearing in the denominator:

$$\lim_{x \to -\infty} \frac{x^2+1}{x+2} \left[\text{divide by } x \right]$$

$$= \lim_{x \to -\infty} \frac{x + \frac{1}{x}}{1 + \frac{2}{x}} = \frac{\lim\limits_{x \to -\infty} x + 0}{1+0} = -\infty .$$

Limits at infinity of rational function: summary

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and $Q(x) = b_m x^m + \cdots + b_1 x + b_0$ be polynomials of degree n and m respectively. Then

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \begin{cases} \dfrac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ \infty \text{ or } -\infty, & \text{if } n > m \end{cases}$$

Limits involving irrational expressions

Example. Evaluate $\lim\limits_{x\to\infty} (\sqrt{x^2+x} - x)$.

Solution. We can not apply the rule for the limit of the difference $\sqrt{x^2+x} - x$, since $\lim\limits_{x\to\infty} \sqrt{x^2+x} = \infty$ and $\lim\limits_{x\to\infty} x = \infty$.

Warning: It is incorrect to write $\lim\limits_{x\to\infty} (\sqrt{x^2+x} - x) = \infty - \infty = 0$.

Here is a trick which will help you to find limits at infinity involving irrationalities:

Rationalize the expression by multiplying (and dividing) by the conjugate expression.

$$\lim_{x\to\infty} (\sqrt{x^2+x} - x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x} = \lim_{x\to\infty} \frac{(\sqrt{x^2+x})^2 - x^2}{\sqrt{x^2+x} + x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x} \left[\text{divide by } x\right] = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.$$

The squeeze theorem for limits at infinity

Example. Calculate $\lim\limits_{x\to\infty} \dfrac{\cos^3 x}{x}$.

Solution. The quotient rule is not applicable here, since $\cos^3 x$ has no limit as $x \to \infty$.

We know that $-1 \le \cos x \le 1$. Therefore, $-1 \le \cos^3 x \le 1$. Multiply all sides of this last inequality by $\dfrac{1}{x}$ (observe that $\dfrac{1}{x} > 0$ as $x \to \infty$):

$$-\frac{1}{x} \le \frac{\cos^3 x}{x} \le \frac{1}{x}.$$ Apply the squeeze theorem:

$$\begin{array}{ccc} -\frac{1}{x} & \frac{\cos^3 x}{x} & \frac{1}{x} \\ {\scriptstyle x\to\infty}\big\downarrow & {\scriptstyle x\to\infty}\big\downarrow & \big\downarrow{\scriptstyle x\to\infty} \\ 0 & 0 & 0 \end{array}$$

Therefore, $\lim\limits_{x\to\infty} \dfrac{\cos^3 x}{x} = 0$.

Application to drawing graphs

Example. Draw the graph of $y = \dfrac{x^2}{x^2-4}$.

Solution.

• Domain: $x^2 - 4 \neq 0$, that is $x \neq \pm 2$.

• Symmetry: the function is **even**, since $y(-x) = \dfrac{(-x)^2}{(-x)^2-4} = \dfrac{x^2}{x^2-4} = y(x)$.
Therefore, its graph is symmetric about the y-axis.

• Vertical asymptotes/behavior as $x \to \pm 2$:

$$\lim_{x\to 2^+} \frac{x^2}{x^2-4} = \left[\ \frac{4}{0^+}\ \right] = \infty, \quad \lim_{x\to 2^-} \frac{x^2}{x^2-4} = \left[\ \frac{4}{0^-}\ \right] = -\infty.$$

Also, $\lim\limits_{x\to -2^+} \dfrac{x^2}{x^2-4} = \left[\dfrac{4}{0^-}\right] = -\infty$ and $\lim\limits_{x\to -2^-} \dfrac{x^2}{x^2-4} = \left[\dfrac{4}{0^+}\right] = \infty$.

Therefore, the lines $x = 2$ and $x = -2$ are the vertical asymptotes of the function.

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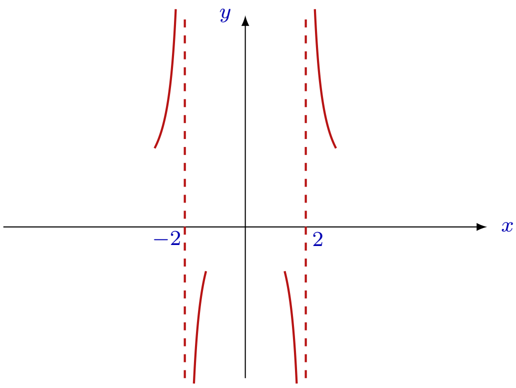
Graphing: vertical asymptotes

Let us place in the coordinate system the information which we have obtained so far:



$$\lim_{x\to 2^+} \frac{x^2}{x^2-4} = \infty$$
$$\lim_{x\to 2^-} \frac{x^2}{x^2-4} = -\infty$$
$$\lim_{x\to -2^+} \frac{x^2}{x^2-4} = -\infty$$
$$\lim_{x\to -2^-} \frac{x^2}{x^2-4} = \infty$$

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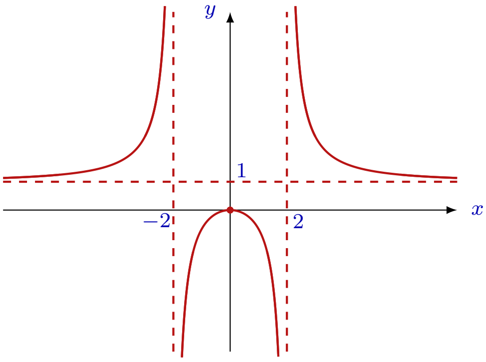
Graphing: horizontal asymptotes

- Horizontal asymptotes/behavior as $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2-4} = \lim_{x\to\pm\infty} \frac{1}{1-\frac{4}{x^2}} = \frac{1}{1-0} = 1^+ .$$

Therefore, the line $y = 1$ is the horizontal asymptote for $y = \dfrac{x^2}{x^2-4}$,
and the graph approaches it **from above:**



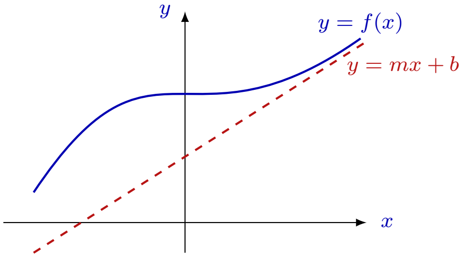
$$y(0) = \frac{0^2}{0^2-4} = 0$$

Oblique asymptotes

Example. $f(x) = \dfrac{2x^2+1}{x} = 2x + \dfrac{1}{x} \approx 2x$ for large x.

This means that the graph of $y = \dfrac{2x^2+1}{x}$ approaches the non-horizontal straight line $y = 2x$ as $x \to \infty$.

Definition. The line $y = mx + b$ (where $m \neq 0$)
is an *oblique* (or *slant*) *asymptote* for the graph of $y = f(x)$
if $\lim_{x\to\infty} (f(x) - (mx+b)) = 0$ or $\lim_{x\to-\infty} (f(x) - (mx+b)) = 0$ or both.



If $y = f(x)$ has the oblique asymptote $y = mx + b$ as $x \to \infty$, then

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and}$$

$$b = \lim_{x\to\infty} (f(x) - mx) .$$

(Similar formulas for oblique asymptotes at $-\infty$)

Oblique asymptotes: example

Example. Find all vertical, horizontal and oblique asymptotes for the graph of the function $f(x) = \dfrac{2x^2+x+1}{x}$. Draw the graph.

Solution. The function has a discontinuity at $x = 0$. Let us calculate

$\displaystyle\lim_{x\to 0} \frac{2x^2+x+1}{x} = \lim_{x\to 0}\left(2x+1+\frac{1}{x}\right)$. This limit doesn't exist, but

$\displaystyle\lim_{x\to 0+} \frac{2x^2+x+1}{x} = \lim_{x\to 0+}\left(2x+1+\frac{1}{x}\right) = \infty$ and

$\displaystyle\lim_{x\to 0-} \frac{2x^2+x+1}{x} = \lim_{x\to 0-}\left(2x+1+\frac{1}{x}\right) = -\infty$.

Therefore, the line $x = 0$ (the y-axis) is the **vertical** asymptote.

For a horizontal asymptote, we calculate

$\displaystyle\lim_{x\to \pm\infty} \frac{2x^2+x+1}{x} = \lim_{x\to \pm\infty}\left(2x+1+\frac{1}{x}\right) = \pm\infty$, not a real number.

Therefore, there are **no** horizontal asymptotes.

Oblique asymptotes: example

To find the oblique asymptote $y = mx + b$ to the graph of $f(x) = \dfrac{2x^2+x+1}{x}$, calculate

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2x^2+x+1}{x^2} = \lim_{x\to\infty}\left(2+\frac{1}{x}+\frac{1}{x^2}\right) = 2\,.$$

$$b = \lim_{x\to\infty} (f(x) - mx) = \lim_{x\to\infty}\left(\frac{2x^2+x+1}{x} - 2x\right)$$

$$= \lim_{x\to\infty}\left(\frac{2x^2+x+1-2x^2}{x}\right) = \lim_{x\to\infty}\left(\frac{x+1}{x}\right) = \lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1\,.$$

Therefore, the oblique asymptote as $x \to \infty$ is $y = 2x + 1$.

Similar calculations for $x \to -\infty$ give the same oblique asymptote.

Oblique asymptotes: example

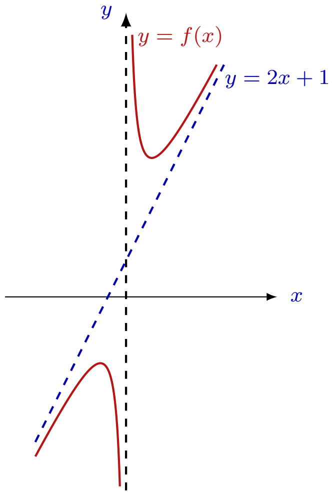
We have found that

The y-axis is a vertical asymptote:
$\lim_{x\to 0^+} f(x) = \infty$,

$\lim_{x\to 0^-} f(x) = -\infty$.

The line $y = 2x + 1$
is an oblique asymptote as $x \to \pm\infty$.

Let us put this information on the graph:



Summary

In this lecture, we discussed

- **limits at infinity**: $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$
- **horizontal asymptotes** to the graphs
- how to calculate limits at infinity for **rational functions** and expressions involving **radicals**
- how to apply the **squeeze theorem** to calculate limits at infinity
- what **oblique asymptotes** are and how to find them

Comprehension checkpoint

• Find the following limits:

$$\lim_{x\to\infty}\frac{1}{1-x},\quad \lim_{x\to-\infty}\frac{1}{x^2-3},\quad \lim_{x\to 0^-}\ln e^{1/x},\quad \lim_{x\to\infty}\arctan x^3$$

• How many horizontal asymptotes may the graph of a function have?

• Draw the graph of a function having the oblique asymptote $y = -x + 5$ as $x \to -\infty$, the horizontal asymptote $y = 1$ as $x \to \infty$, and the vertical asymptote $x = 2$.

• Can the graph of an odd function have the asymptote $y = 1$ as $x \to \infty$ and the asymptote $y = -2$ as $x \to -\infty$?