Lecture 8

Infinite Limits

Objectives

There are two types of limits involving infinity:

- infinite limits $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} f(x) = -\infty$ and
- limits at infinity $\lim_{x \to \infty} f(x) = L$, $\lim_{x \to -\infty} f(x) = L$.

In this lecture we discuss **infinite limits**

and applications to finding **vertical asymptotes** of the graphs of functions.

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Infinite limits

Definition. Let $f(x)$ be a function and a be a number.
One says that f *approaches infinity as* x *approaches* a
if $f(x)$ is arbitrary large whenever x is sufficiently close to a (but not equal to a).

Notations: $\lim_{x \to a} f(x) = \infty$ or $f(x) \underset{x \to a}{\longrightarrow} \infty$.

Similarly, we say that f *approaches negative infinity as* x *approaches* a
i if $f(x)$ takes arbitrary large negative values
whenever x is sufficiently close to a (but not equal to a):

Notations: $\lim_{x \to a} f(x) = -\infty$ or $f(x) \underset{x \to a}{\longrightarrow} -\infty$.

One-sided limits

$\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^-} f(x) = \infty$, $\lim_{x \to a^+} f(x) = -\infty$, $\lim_{x \to a^-} f(x) = -\infty$

are defined in a similar way.

⚠ **Warning:** ∞ and $-\infty$ are **not** real numbers, and can not be treated as real numbers.

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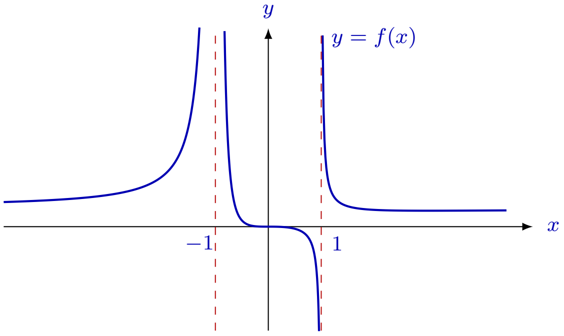
Finding limits from graphs

Here is the graph of a function.



The following limits can be read off the graph:

$\lim_{x \to -1^-} f(x) = \infty$ $\lim_{x \to -1^+} f(x) = \infty$ $\lim_{x \to -1} f(x) = \infty$

$\lim_{x \to 1^-} f(x) = -\infty$ $\lim_{x \to 1^+} f(x) = \infty$ $\lim_{x \to 1} f(x)$ does not exist (DNE)

Vertical asymptotes

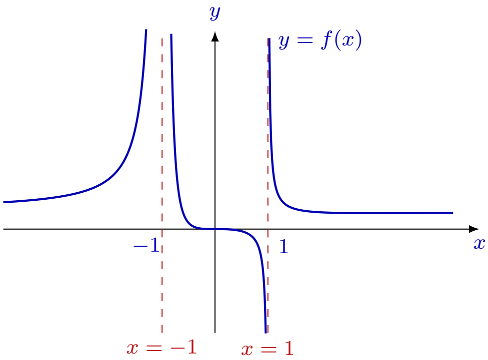
The vertical line $x = a$ is called a *vertical asymptote* for the graph of $y = f(x)$ if at least one of the following holds true:

$\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a^+} f(x) = -\infty$ $\lim_{x \to a^-} f(x) = -\infty$

Example.

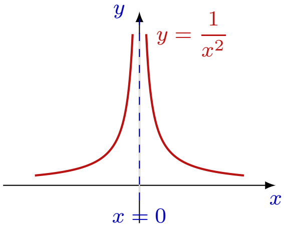
The graph of the function $y = f(x)$ shown to the right has two **vertical asymptotes**: $x = -1$ and $x = 1$.



Reciprocal functions

We now discuss the standard examples of infinite limits.

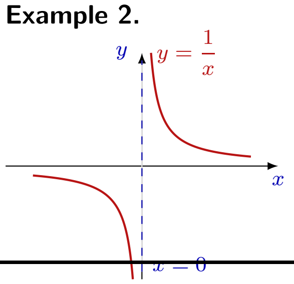
Example 1.



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$x = 0$ is a **vertical asymptote** for $y = \frac{1}{x^2}$.

Example 2.

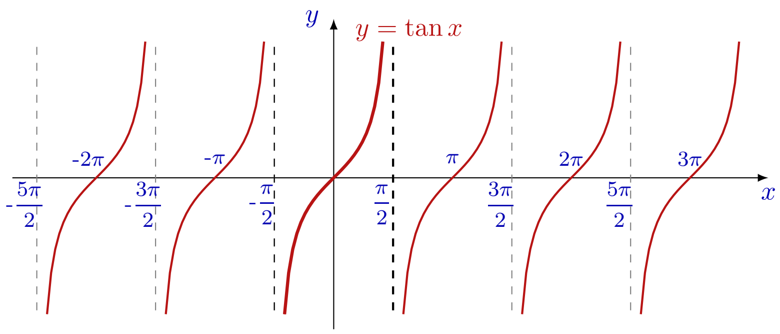


$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \to 0} \frac{1}{x} \text{ DNE.}$$

$x = 0$ is a **vertical asymptote** for $y = \frac{1}{x}$.

The tangent function

Example 3.



$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty, \quad \lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty, \quad \lim_{x \to \frac{\pi}{2}} \tan x \text{ DNE.}$$

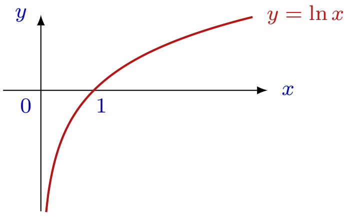
$x = \frac{\pi}{2}$ is a **vertical asymptote** for $y = \tan x$.

$y = \tan x$ has infinitely many vertical asymptotes. They are $x = \frac{\pi}{2} + \pi n, \ n \in \mathbb{Z}$.

The logarithmic function

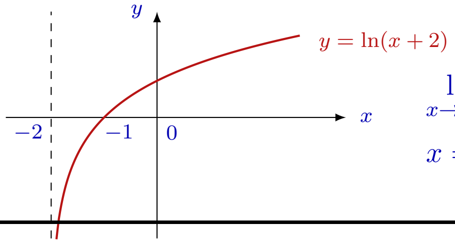
Example 4. Calculate $\lim_{x\to 0^+} \ln x\,,\quad \lim_{x\to -2^+} \ln(x+2)$.

Solution. Draw and see:



$$\lim_{x\to 0^+} \ln x = -\infty$$

$x=0$ is a vertical asymptote of the graph.



$$\lim_{x\to -2^+} \ln(x+2) = -\infty$$

$x=-2$ is a vertical asymptote.

Rational functions

Example 5. Evaluate $\lim_{x\to 2^+} \dfrac{3}{x-2}$ and $\lim_{x\to 2^-} \dfrac{3}{x-2}$.

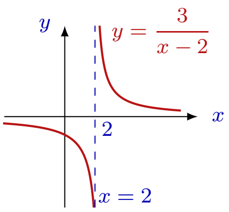
Solution. When x approaches 2 *from the right*, $x-2$ approaches 0 *from the right*. Observe that $x-2>0$ since $x>2$.

So $x-2 \underset{x\to 2^+}{\longrightarrow} 0^+$. Therefore, $\dfrac{1}{x-2} \underset{x\to 2^+}{\longrightarrow} \infty$ and $\lim_{x\to 2^+} \dfrac{3}{x-2} = \infty$.

When x approaches 2 *from the left*, $x-2$ approaches 0 *from the left*.

Observe that $x-2<0$ since $x<2$.

So $x-2 \underset{x\to 2^-}{\longrightarrow} 0^-$. Therefore, $\dfrac{1}{x-2} \underset{x\to 2^-}{\longrightarrow} -\infty$ and $\lim_{x\to 2^-} \dfrac{3}{x-2} = -\infty$.



Here is a convenient informal way to write down our calculations:

$$\lim_{x\to 2^+} \frac{3}{x-2} = \left[\frac{3}{0^+}\right] = \infty \quad\text{and}\quad \lim_{x\to 2^-} \frac{3}{x-2} = \left[\frac{3}{0^-}\right] = -\infty.$$

Finding vertical asymptotes

Example 1. Find all vertical asymptotes of the graph of the rational function $f(x) = \dfrac{x^2}{(x^2-1)(x+2)}$.

Solution. We inspect the points where the denominator vanishes.
Only at these points can a rational functions have vertical asymptotes.

$(x^2-1)(x+2)=0 \iff (x-1)(x+1)(x+2)=0 \quad \iff x=1$ or $x=-1$ or $x=-2.$

We now calculate one-sided limits to verify the existence of vertical asymptotes at $x=1,\ -1,\ -2.$

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Calculating limits

$$\lim_{x\to1^+}\frac{x^2}{(x^2-1)(x+2)}=\frac{1^2}{((1^+)^2-1)(1+2)}=\left[\frac{1}{0^+\cdot3}\right]=\infty$$

$$\lim_{x\to1^-}\frac{x^2}{(x^2-1)(x+2)}=\frac{1^2}{((1^-)^2-1)(1+2)}=\left[\frac{1}{0^-\cdot3}\right]=-\infty$$

$$\lim_{x\to-1^+}\frac{x^2}{(x^2-1)(x+2)}=\frac{(-1)^2}{((-1^+)^2-1)(-1+2)}=\left[\frac{1}{0^-\cdot3}\right]=-\infty$$

$$\lim_{x\to-1^-}\frac{x^2}{(x^2-1)(x+2)}=\frac{(-1)^2}{((-1^-)^2-1)(-1+2)}=\left[\frac{1}{0^+\cdot3}\right]=\infty$$

$$\lim_{x\to-2^+}\frac{x^2}{(x^2-1)(x+2)}=\frac{(-2)^2}{((-2)^2-1)(-2^++2)}=\left[\frac{4}{3\cdot0^+}\right]=\infty$$

$$\lim_{x\to-2^-}\frac{x^2}{(x^2-1)(x+2)}=\frac{(-2)^2}{((-2)^2-1)(-2^-+2)}=\left[\frac{4}{3\cdot0^-}\right]=-\infty$$

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Graphing vertical asymptotes

We have calculated the following limits for $f(x) = \dfrac{x^2}{(x^2-1)(x+2)}$:

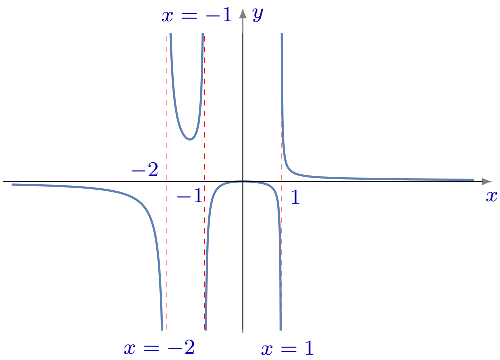
$\lim_{x\to 1^+} f(x) = \infty$, $\quad \lim_{x\to 1^-} f(x) = -\infty$,

$\lim_{x\to -1^+} f(x) = -\infty$, $\quad \lim_{x\to -1^-} f(x) = \infty$,

$\lim_{x\to -2^+} f(x) = \infty$, $\quad \lim_{x\to -2^-} f(x) = -\infty$.

Therefore the lines $x = 1$, $x = -1$ and $x = -2$ are the vertical asymptotes.

This information can help us to sketch the graph of $f(x)$



Finding vertical asymptotes (cont.)

Example 2. Find all vertical asymptotes to the graph of $f(x) = \dfrac{x+1}{x^2-1}$.

Solution. We inspect the points where the denominator vanishes. Only at these points can a rational function have vertical asymptotes.

$x^2 - 1 = 0 \iff (x-1)(x+1) = 0 \quad \iff x = 1$ or $x = -1$.

We now calculate limits to verify the existence of vertical asymptotes at $x = 1, -1$.

$$\lim_{x\to 1^+} \frac{x+1}{x^2-1} = \frac{1+1}{(1^+)^2-1} = \left[\frac{2}{0^+}\right] = \infty$$

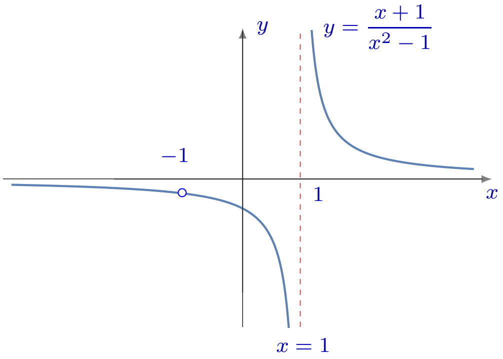
$$\lim_{x\to 1^-} \frac{x+1}{x^2-1} = \frac{1+1}{(1^-)^2-1} = \left[\frac{2}{0^-}\right] = -\infty$$

$$\lim_{x\to -1} \frac{x+1}{x^2-1} = \lim_{x\to -1} \frac{x+1}{(x-1)(x+1)} = \lim_{x\to -1} \frac{1}{x-1} = -\frac{1}{2} \neq \pm\infty.$$

Therefore, $x = 1$ is a vertical asymptote, while $x = -1$ is not.

Graphing asymptotes

We have established that the graph of $f(x) = \dfrac{x+1}{x^2-1}$ has only the vertical asymptote $x = 1$.



Notice that the function is not defined at $x = 1$ and $x = -1$.
At $x = 1$ the graph has a vertical asymptote, and at $x = -1$ the graph has a hole.

Summary

In this lecture we studied

• **infinite limits**: $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$

and one-sided **infinite limits**:

$\lim_{x \to a^-} f(x) = \infty$ or $-\infty$ and $\lim_{x \to a^+} f(x) = \infty$ or $-\infty$

• the notion of a vertical asymptote

• appearances of the graphs of some standard functions with vertical asymptotes

• how to find vertical asymptotes of rational functions

Comprehension checkpoint

• Let $\lim_{x\to 0^-} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = -\infty$.

Is it true that the line $x = 0$ is a vertical asymptote for the graph of $y = f(x)$?

• Sketch the graph of a function $y = f(x)$ such that

$\lim_{x\to -2} f(x) = \infty$, $\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = \infty$.

• How would one verify that the line $x = a$ is a vertical asymptote for the graph of $y = f(x)$?

• Let $y = \dfrac{x}{x^2 + x}$. Is it true that $x = 0$ is a vertical asymptote for the graph of this function? Justify your answer.