Lecture 7

Calculation of Limits

Objectives

In this lecture, we will discuss various **techniques** for the calculation of limits:

- Direct substitution
- Algebraic transformations
- The squeeze theorem

Direct substitution

Direct substitution is used to calculate the limit of a function at a point where the function is **continuous**.

In this case, the limit is equal to the value of the function:

$$\lim_{x \to a} f(x) = f(a)$$

Remember that elementary functions are continuous where defined.

Example 1. $\lim_{x \to 0} \frac{3x-1}{x^2+1} = \frac{3 \cdot 0 - 1}{0^2+1} = -1$

$$\lim_{x \to \pi} \frac{\cos x}{\sin^2 x - 1} = \frac{\cos \pi}{\sin^2 \pi - 1} = \frac{-1}{(0^2) - 1} = 1$$

$$\lim_{x \to 1} e^{x \ln x} = e^{1 \cdot \ln 1} = e^0 = 1$$

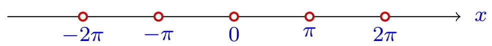
$$\lim_{x \to -1} \frac{|x+1|}{x-1} = \frac{|-1+1|}{-1-1} = \frac{|0|}{-2} = 0$$

Continuity

Example 2. Find the intervals of continuity of the function $f(x) = \dfrac{x}{\sin x}$.

Evaluate $\lim\limits_{x \to \pi/2} \dfrac{x}{\sin x}$.

Solution. f is an elementary function and it is continuous on its domain.
f is defined for all x where $\sin x \neq 0$, that is for $x \neq n\pi$, $n \in \mathbb{Z}$.



Therefore, the domain of f is the union of infinitely many open intervals:

$$\ldots \cup (-2\pi, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots = \bigcup_{n \in \mathbb{Z}} \big(n\pi, (n+1)\pi\big).$$

Therefore, f is continuous on $\bigcup\limits_{n \in \mathbb{Z}} \big(n\pi, (n+1)\pi\big)$.

Since f is continuous at $x = \pi/2$, the limit is calculated by **direct substitution**: $\lim\limits_{x \to \pi/2} \dfrac{x}{\sin x} = \dfrac{\pi/2}{\sin \pi/2} = \dfrac{\pi/2}{1} = \dfrac{\pi}{2}$.

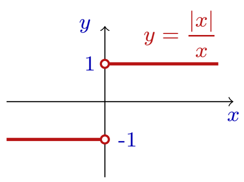
When direct substitution does not work

Example 1. Find $\lim\limits_{x \to 0} \dfrac{|x|}{x}$.

Solution. Direct substitution of 0 into $\dfrac{|x|}{x}$ makes **no** sense: $\dfrac{|0|}{0}$ is not defined.

Let us investigate the behavior of the function $y = \dfrac{|x|}{x}$ near 0.

Since $|x| = \begin{cases} x, \; x \geq 0 \\ -x, \; x < 0, \end{cases}$ we have $\dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & x > 0 \\ \dfrac{-x}{x} = -1, & x < 0. \end{cases}$



$$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Therefore, $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

When direct substitution does not work

Example 2. Calculate $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Solution. Direct substitution of $x = 1$ into $\frac{x^2 - 1}{x - 1}$ does not work: $\frac{1^2 - 1}{1 - 1} = \frac{0}{0}$ is undefined.

The quotient rule for limits $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{\lim_{x \to 1}(x^2 - 1)}{\lim_{x \to 1}(x - 1)}$ does not work either,

since the limit in the denominator is 0: $\lim_{x \to 1}(x - 1) = 0$

What should we do?

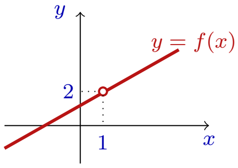
Algebraic transformations

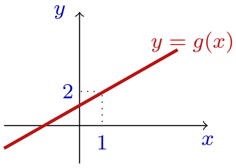
Let's do some algebra:

$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1$, if $x \neq 1$.

We have two functions, $f(x) = \frac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$.

$f(x)$ is not defined for $x = 1$, and $f(x) = g(x)$ for all $x \neq 1$:





Since $f(x) = g(x)$ for all $x \neq 1$, we may use the **substitution of a function** rule:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2.$$

Algebraic transformations for limit calculations

Rationalizing can be used to clear the denominator.

Example 1. Evaluate $\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$.

Solution. Direct substitution of $x=0$ is not good: $\frac{\sqrt{0^2+9}-3}{0^2} = \frac{3-3}{0} = \frac{0}{0}$.
Here we can **rationalize** the numerator:

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to 0} \frac{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)}{x^2(\sqrt{x^2+9}+3)}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+9})^2-3^2}{x^2(\sqrt{x^2+9}+3)} = \lim_{x\to 0} \frac{(x^2+9)-9}{x^2(\sqrt{x^2+9}+3)} = \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+9}+3)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{\sqrt{0^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

Algebraic transformations for limit calculation

Example 2. Evaluate $\lim_{x\to 2} \frac{x^4-16}{x^3-8}$.

Solution. Direct substitution does not work: $\lim_{x\to 2} \frac{x^4-16}{x^3-8} = \frac{2^4-16}{2^3-8} = \frac{0}{0}$ ☹

Try algebra: $x^4-16 = (x^2)^2-4^2 = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4)$.

$x^3-8 = x^3-2^3 = (x-2)(x^2+2x+4)$

☞ $a^3-b^3 = (a-b)(a^2+ab+b^2)$

So in this case we may **simplify** the expression before taking the limit:

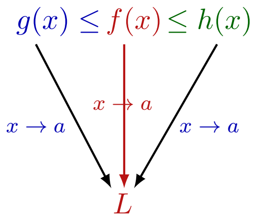
$$\lim_{x\to 2} \frac{x^4-16}{x^3-8} = \lim_{x\to 2} \frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)(x^2+4)}{\cancel{(x-2)}(x^2+2x+4)}$$

$$\lim_{x\to 2} \frac{(x+2)(x^2+4)}{x^2+2x+4} = \frac{(2+2)(2^2+4)}{2^2+2\cdot 2+4} = \frac{4\cdot 8}{4\cdot 3} = \frac{8}{3}$$

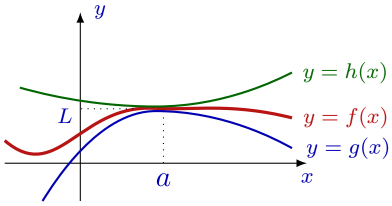
The squeeze theorem

Let $g(x) \le f(x) \le h(x)$ for all x near a (except possibly $x = a$ itself).

If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.



near $x = a$



Other names for the squeeze theorem:

the sandwich theorem

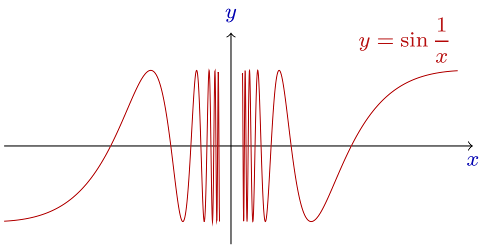
the two policemen theorem

Limit calculations using the squeeze theorem

Example. Calculate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution. The product rule $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

is **not** applicable, since $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist:



Limits calculation using squeeze theorem

Let us estimate the function $x^2 \sin\left(\frac{1}{x}\right)$ from below and from above,
that is, find functions g and h whose limits we know such that

$g(x) \le x^2 \sin\left(\frac{1}{x}\right) \le h(x)$.

We know that $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ any x .

Multiply all terms of this inequality by x^2 (note that $x^2 > 0$):

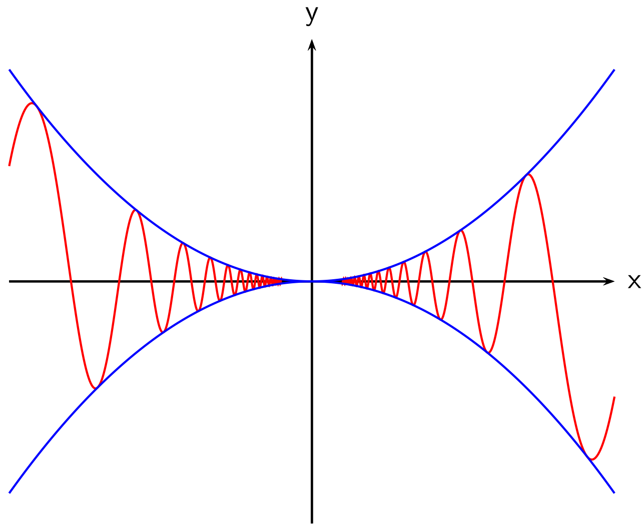
$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2 .$$ So, taking $g(x) = -x^2$, $h(x) = x^2$.

$x \to 0$: $-x^2 \to 0$, $x^2 \sin\left(\frac{1}{x}\right) \to 0$, $x^2 \to 0$

The squeeze theorem implies that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$

The squeeze theorem: illustration

The graph of the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ is **squeezed** between the graphs of $g(x) = -x^2$ and $h(x) = x^2$:



Summary

In this lecture, we learned how to calculate limits by

- **direct substitution**
- **algebraic transformations** leading to clearing the denominator
- **squeeze theorem**

Comprehension checkpoint

- Calculate the limit $\lim_{x \to 3} \dfrac{\sin(x^2 - 9)}{\tan x}$
- Calculate the limit $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$
- Let $\sqrt{4 - 2x^2} \leq f(x) \leq \sqrt{4 - x^2}$ for $-1 \leq x \leq 1$. Find $\lim_{x \to 0} f(x)$.