Lecture 6

Limit and Continuity

Objectives

In this lecture, we'll discuss the following topics:

- The definition of limit
- Properties of limits
- Continuity
- The intermediate value theorem

What are limits about?

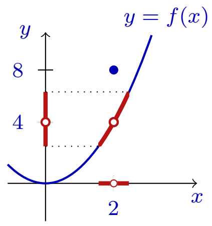
Calculus studies functions.

How do functions behave? What is behavior of a function overall?

What is the local behavior of a function?

The notion of limit describes the behavior of a function near a point, that is, its **local behavior**.

Example. What is behavior of $f(x) = \begin{cases} x^2, & x \neq 2 \\ 8, & x = 2 \end{cases}$ near the point $x = 2$?



Notice that when x is near 2,
but is not equal to 2,
then $f(x)$ is near 4.

We need to make this more **precise**.

The definition of limit

Definition (informal). Let $f(x)$ be a function and a be a number.

A number L is called the *limit of f as x approaches a*,

if $f(x)$ can be made arbitrary close to L by choosing x sufficiently close to a (but not equal to a).

Notation: $\lim_{x \to a} f(x) = L$ or $f(x) \underset{x \to a}{\longrightarrow} L$.

Remarks.

1. $f(a)$, the value of the function f at a, does **not** enter into the definition of the limit.
2. Why is the definition informal?

The expressions "arbitrarily close" and "sufficiently close" are imprecise.

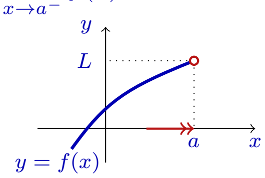
Loosely speaking,
$f(x)$ gets closer and closer to L as x gets closer and closer to a.

3. You will learn a precise definition of limit when you study Analysis.

One-sided limits

• Left-hand limit:

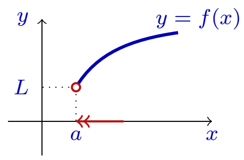
$\lim_{x \to a^-} f(x) = L$ means that $f(x)$ is arbitrarily close to L whenever x is sufficiently close to a and $x < a$.



The limit of f when x approaches a **from the left** is equal to L.

• Right-hand limit:

$\lim_{x \to a^+} f(x) = L$ means that $f(x)$ is arbitrarily close to L whenever x is sufficiently close to a and $x > a$.



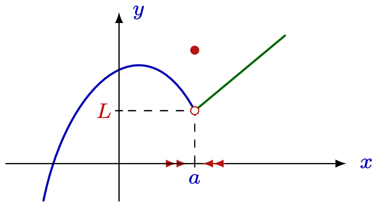
The limit of f when x approaches a **from the right** is equal to L.

Two important facts

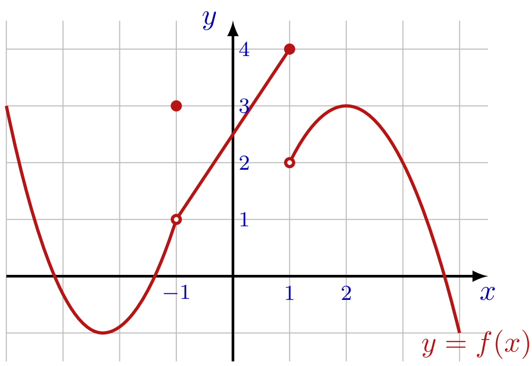
Theorem 1. If a limit exists, then it is **unique**.

Theorem 2. The limit of a function exists if and only if both left- and right-hand limits exist and are equal:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$



Reading limits from a graph



$\lim_{x\to 2} f(x) = 3,\quad \lim_{x\to 2^-} f(x) = 3,\quad \lim_{x\to 2^+} f(x) = 3$

$\lim_{x\to 1^+} f(x) = 2,\quad \lim_{x\to 1^-} f(x) = 4,\quad \lim_{x\to 1} f(x)$ doesn't exist

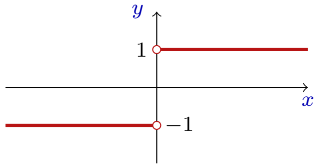
$\lim_{x\to -1^+} f(x) = 1,\quad \lim_{x\to -1^-} f(x) = 1,\quad \lim_{x\to -1} f(x) = 1\,.$

When the limit does not exist

Problem. Find the limit $\lim\limits_{x\to 0} \dfrac{|x|}{x}$.

Solution. Since $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$ we get $\dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x}, & x > 0 \\ \dfrac{-x}{x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$

Here is the graph of our function $y = \dfrac{|x|}{x}$:



We see that $\lim\limits_{x\to 0+} \dfrac{|x|}{x} = \lim\limits_{x\to 0+} 1 = 1$, and $\lim\limits_{x\to 0-} \dfrac{|x|}{x} = \lim\limits_{x\to 0-} (-1) = -1$.

So the right- and left-hand limits do not coincide.

Therefore, $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist.

Properties of limits (limit laws)

The following limit laws will be proven in Analysis.
They are grouped here for reference.

Let $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Then

- **Limit of a sum:** $\lim\limits_{x\to a} \Big(f(x) + g(x)\Big) = L + M$
- **Limit of a difference:** $\lim\limits_{x\to a} \Big(f(x) - g(x)\Big) = L - M$
- **Limit of a product:** $\lim\limits_{x\to a} f(x)g(x) = LM$
- **Limit of a multiple:** $\lim\limits_{x\to a} cf(x) = cL$
- **Limit of a quotient:** $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$

Properties of limits (limit laws)

- **Limit of a composition:**

If $\lim_{x\to a} f(x) = L$ and $\lim_{y\to L} g(y) = M$, then $\lim_{x\to a} g(f(x)) = M$.

- **Inequality and limits:**

If $f(x) \le g(x)$ near a and $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, then $L \le M$.

- **Limit of a constant function:** $\lim_{x\to a} c = c$.

- **Substitution of a number:** $\lim_{x\to a} x = a$.

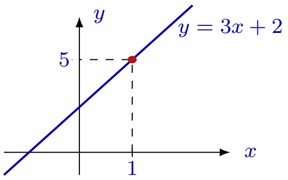
- **Substitution of a function:**

If $f(x) = g(x)$ for all $x \ne a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Simplest examples

Example 1. Calculate the limit $\lim_{x\to 1}(3x+2)$.

Solution. $\lim_{x\to 1}(3x+2) = \lim_{x\to 1}(3x) + \lim_{x\to 1} 2 = 3\lim_{x\to 1} x + 2 = 3\cdot 1 + 2 = 5.$



Example 2. Calculate the limit $\lim_{x\to \pi/2} e^{\sin x}$.

Solution. We use the rule for the limit of a composition.

Since $\lim_{x\to\pi/2} \sin x = \sin\frac{\pi}{2} = 1$, and $\lim_{y\to 1} e^y = e^1 = e$,

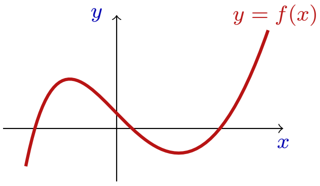
we have $\lim_{x\to\pi/2} e^{\sin x} = e$.

Continuity

Definition. A function $f(x)$ is *continuous* at point a if $\lim\limits_{x \to a} f(x) = f(a)$.

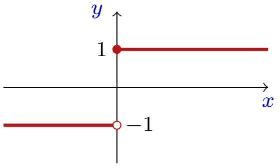
f is *continuous on an interval* if $f(x)$ is continuous at all points of the interval.

Example 1.



This function is continuous at all points.
The graph of a continuous function can be drawn **without** taking the pen off the paper.

Example 2.



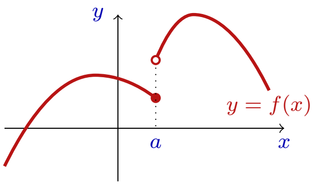
The function $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ is **not** continuous at 0,
since $\lim\limits_{x \to 0} f(x)$ doesn't exist.

One says that $f(x)$ has a **discontinuity** at 0.

Discontinuity

A function is *discontinuous* at a point, if it is **not** continuous at that point.

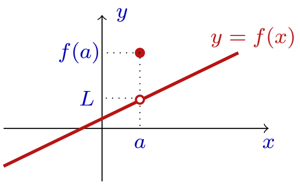
Example 1.



This function is **discontinuous** at point a,
since $\lim\limits_{x \to a} f(x)$ doesn't exist.

The graph of a discontinuous function can't be drawn **without** taking the pen off the paper.

Example 2.



This function is **discontinuous** at point a.
The limit $\lim\limits_{x \to a} f(x)$ exists,
but is not equal to the value of f at a:
$\lim\limits_{x \to a} f(x) = L \neq f(a)$.

Elementary functions are continuous where defined

Theorem. All **elementary functions**
(power, exponential, logarithmic, trigonometric, inverse trigonometric,
and their sums, differences, products, quotients, and compositions)
are **continuous** where they are defined. Therefore,
limits of elementary functions can be evaluated by **direct substitution**.

Example 1. Where is $f(x) = \dfrac{\sqrt{x}}{x-1}$ continuous? Find $\lim\limits_{x\to 4} \dfrac{\sqrt{x}}{x-1}$.

Solution. $f(x)$ is an **elementary** function, so it is continuous where it is defined.
The domain of f is the set of all x such that $x \geq 0$, $x \neq 1$.
Therefore, f is continuous on $[0,1) \cup (1,\infty)$.

Since $x = 4$ is in the domain, f is continuous at $x = 4$
and the limit can be calculated by **direct substitution**:

$$\lim_{x\to 4} \frac{\sqrt{x}}{x-1} = \frac{\sqrt{4}}{4-1} = \frac{2}{3}.$$

Making a piecewise-defined function continuous

Example 2. Find the value of a constant a for which the function $f(x) = \begin{cases} x^2-1, & x \leq 2 \\ ax+4, & x > 2 \end{cases}$
is continuous for **all** $x \in \mathbb{R}$.

Solution. For $x < 2$, $f(x) = x^2 - 1$, so f is continuous for $x < 2$.

For $x > 2$, $f(x) = ax + 4$, so f is continuous for $x > 2$.

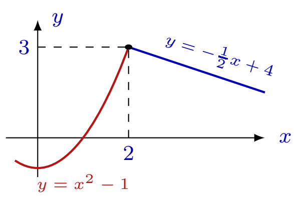
Therefore, $f(x)$ is continuous for all $x \neq 2$ regardless of a.

We have to choose a in such a way that $f(x)$ will also be continuous at $x = 2$.
This will be the case if $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$. Since

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2-1) = x^2 - 1\Big|_{x=2} = 3 \text{ and}$$

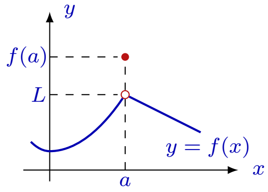
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax+4) = ax + 4\Big|_{x=2} = 2a + 4,$$

we should have $3 = 2a + 4 \iff \boxed{a = -1/2}$

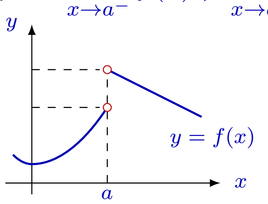


Types of discontinuities

1. A function $f(x)$ has a *removable* discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists, but is not equal to the value of the function at the point: $\lim_{x \to a} f(x) \neq f(a)$.



2. A function $f(x)$ has a *jump* discontinuity at $x = a$ if both one-sided limits exist but are not equal: $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.



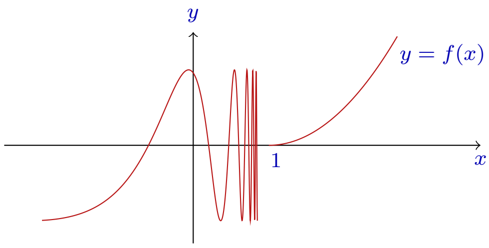
Types of discontinuities

3. A function $f(x)$ has an *essential* discontinuity at $x = a$ if one of one-sided limits does not exist.

For example, the function $f(x) = \begin{cases} \sin \frac{1}{x-1}, & x < 1 \\ (x-1)^2, & x \geq 1 \end{cases}$

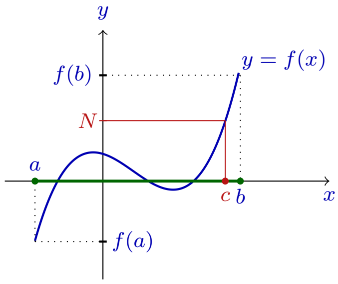
has an essential discontinuity at $x = 1$ since

$\lim_{x \to 1^-} f(x)$ does not exist. Notice that $\lim_{x \to 1^+} f(x) = 0$:

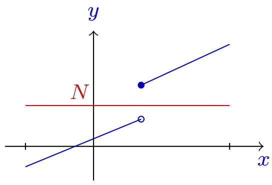


The intermediate value theorem

Theorem. Let f be a **continuous** function on the closed interval $[a, b]$. Suppose $f(a) \neq f(b)$.
Then for any number N between $f(a)$ and $f(b)$
there exists $c \in (a, b)$ such that $f(c) = N$.



f must be continuous,
otherwise the theorem doesn't hold true:



An application of the intermediate value theorem

The intermediate value theorem may help in finding roots of equations.

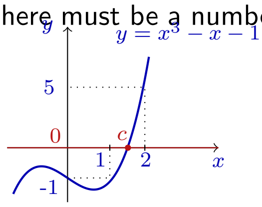
Problem. Show that the equation $x^3 - x - 1 = 0$ has a root in the interval $[1, 2]$.

Solution. We are going to apply the intermediate value theorem
to the continuous function $f(x) = x^3 - x - 1$ on the closed interval $[1, 2]$.

We calculate $f(1)$ and $f(2)$:

$$f(1) = 1^3 - 1 - 1 = -1, \quad f(2) = 2^3 - 2 - 1 = 5.$$

Since 0 is between -1 and 5, the intermediate value theorem states that
that there must be a number $c \in [1, 2]$ such that $f(c) = 0$. This number c is a root of the equation.



Summary

In this lecture we studied

- the **limit** of a function at a point: $\lim_{x\to a} f(x)$
- **one-sided limit** of a function at a point:

$$\lim_{x\to a^-} f(x), \quad \lim_{x\to a^+} f(x),$$

- limit laws
- **continuity** of a function
- **discontinuities** and their types
- the **intermediate value theorem**

Comprehension check

- Is it true that if $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1^-} f(x) = 2$ then $\lim_{x\to 1^+} f(x) = 2$?
- Sketch the graph of a function $y = f(x)$ having the following properties:

$\lim_{x\to 0} f(x) = 1, \quad \lim_{x\to 2^-} f(x) = 3, \quad \lim_{x\to 2^+} f(x) = -1.$

- Find the limit $\lim_{x\to 0} \ln(\cos x)$.
- Is the following reasoning correct:

$\lim_{x\to 0} \frac{x}{x} = \frac{0}{0} = 0?$

- Draw the graph of the function $y = \frac{x}{x}$ and explain how to calculate $\lim_{x\to 0} \frac{x}{x}$.