Lecture 5

Elementary Functions. Part 3

Objectives

What are elementary functions?

Power, exponential, logarithmic, trigonometric, inverse trigonometric functions

and their sums, differences, products, quotients, and compositions.

In this lecture, we review

- logarithmic functions ($y = \log_a x$) as inverse for exponential functions $y = a^x$
- inverse trigonometric functions

($y = \arcsin x,\ y = \arccos x,\ y = \arctan x$).

Inverse functions

Recall that a function $f : D \to R$ with domain D and range R
is called *invertible* if there exists an *inverse* function $f^{-1} : R \to D$
with domain R and range $D,$ which has the following properties:
$f^{-1}(f(x)) = x$ for any $x \in D$
and $f(f^{-1}(y)) = y$ for any $y \in R$.

In Lecture 4 we proved the following important result:

If a function is *monotonic* on an interval
(that is, if it is strictly increasing or strictly decreasing on the interval),
then the function is invertible on that interval.

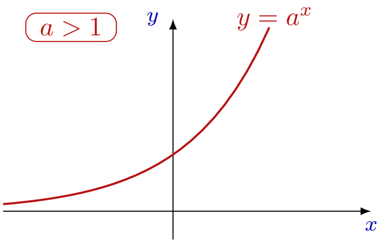
We also showed that the graphs of a function and its inverse
are **symmetric** to each other about the line $y = x.$

Exponential functions are monotonic

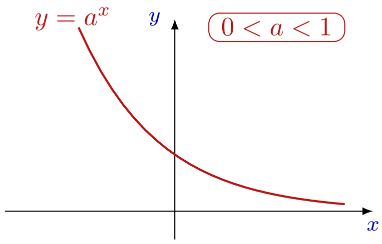
Let $f(x) = a^x$ be an exponential function with $a \neq 1$.

The domain of f is the whole real axis $(-\infty, \infty)$, its range is $(0, \infty)$.

If $a > 1$, then $f(x) = a^x$ increases on the whole domain:



If $0 < a < 1$, then $f(x) = a^x$ decreases on the whole domain:



In either case, $f(x) = a^x$ is **monotonic** on $(-\infty, \infty)$.

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Logarithms as inverses of exponentials

Since $f(x) = a^x$ is monotonic, it is invertible.

Its inverse is called the *logarithm* function with base a .

The inverse of the exponential $f(x) = a^x$ is denoted by $f^{-1}(y) = \log_a(y)$.

The domain of the logarithm function is the range of the exponential, that is, $(0, \infty)$.

The range of the logarithm function is the domain of the exponential, that is, $(-\infty, \infty)$.

By the definition of inverse function,

$f^{-1}(f(x)) = x$ for any $x \in \mathbb{R}$, and $f(f^{-1}(y)) = y$ for any $y \in (0, \infty)$.

That is,

$\log_a(a^x) = x$ for any $x \in \mathbb{R}$, and $a^{\log_a y} = y$ for any $y \in (0, \infty)$.

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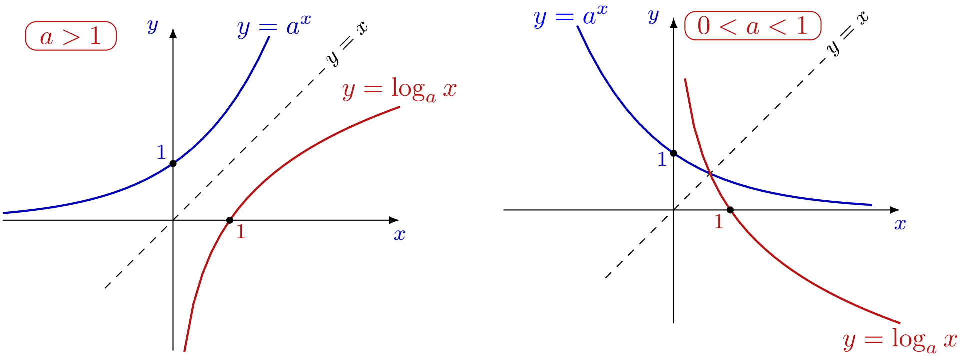
Graphs of a logarithm function

The graphs of a function and its inverse are symmetric about the line $y = x$.

Therefore, the graph of $y = \log_a x$ is obtained from the graph of $y = a^x$ by reflection in the line $y = x$:



The laws of logarithms

The properties (laws) of logarithms follow from the properties (laws) of exponents.

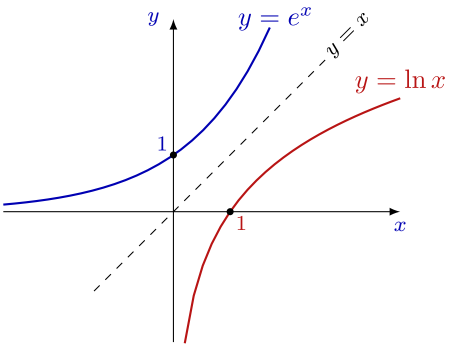
Logarithm	Exponent
$\log_a 1 = 0$	$a^0 = 1$
$\log_a a = 1$	$a^1 = a$
$\log_a(xy) = \log_a x + \log_a y$	$a^{x+y} = a^x \cdot a^y$
$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$a^{x-y} = \frac{a^x}{a^y}$
$\log_a(x^b) = b \log_a x$	$(a^x)^y = a^{xy}$
$\log_a x = \frac{\log_b x}{\log_b a}$	$\left(a^{\frac{1}{y}}\right)^x = a^{\frac{x}{y}}$

The natural logarithm

The *natural* logarithm is the logarithm with base e.

It is denoted by $\ln x$: $\ln x = \log_e x$.

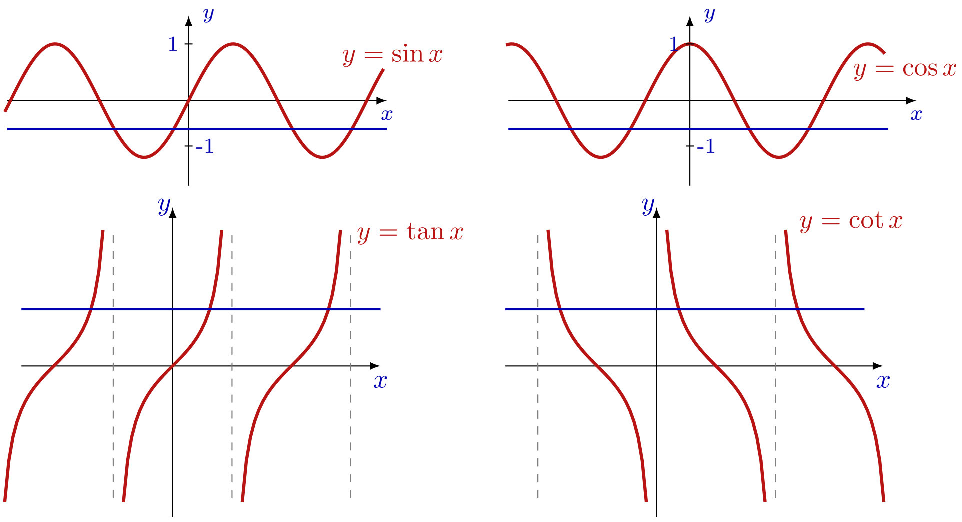
The natural logarithm is the king in the land of the logarithms.



Any logarithm may be expressed in terms of the natural logarithm: $\log_a x = \dfrac{\ln x}{\ln a}$

Trigonometric functions are not invertible

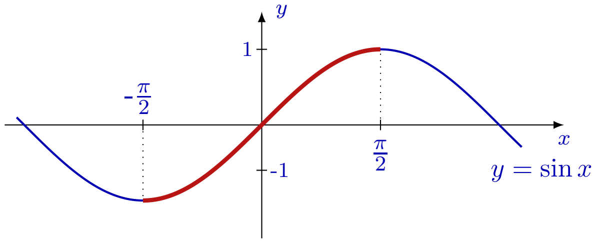
The trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$ are **not** invertible, since they all fail the horizontal line test:



The sine function with a restricted domain

$y = \sin x$ is not invertible on its domain $(-\infty, \infty)$.

Let us restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:



that is, consider the function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by the formula $f(x) = \sin x$ for each $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

This function has domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$.

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The inverse sine

The function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$, defined by the formula $f(x) = \sin x$ for each $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, increases on its domain. Therefore, it is invertible. The inverse is called the *inverse sine*, or *arcsine*.

The inverse is denoted by $\arcsin(y)$.

The domain of the arcsine is the range of the sine, that is, $[-1, 1]$.
The range of the arcsine is the domain of the sine, that is, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

By the definition of inverse function,

$\arcsin(\sin x) = x$ for any $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and

$\sin(\arcsin y) = y$ for any $y \in [-1, 1]$.

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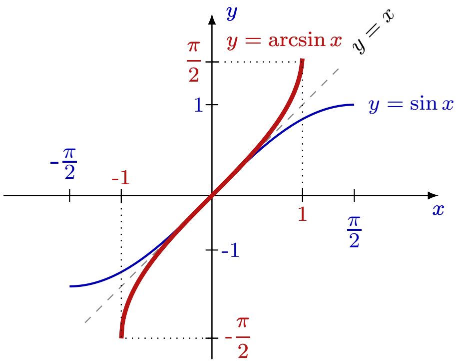
The graph of the inverse sine function

The graphs of a function and its inverse are symmetric about the line $y = x$.

Therefore, the graph of $y = \arcsin x$ is obtained from the graph of $y = \sin x$ for $x \in [-\pi/2, \pi/2]$, by the reflection in the line $y = x$:



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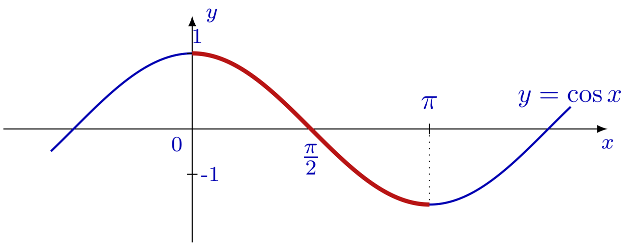
The cosine function with a restricted domain

$y = \cos x$ is not invertible on its domain $(-\infty, \infty)$.

Let us restrict the domain to the interval $[0, \pi]$:



that is, consider the function $f : [0, \pi] \to [-1, 1]$ defined by the formula $f(x) = \cos x$ for $x \in [0, \pi]$.

This function has domain $[0, \pi]$ and range $[-1, 1]$.

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The inverse cosine

The function $f : [0, \pi] \to [-1, 1]$, defined by the formula $f(x) = \cos x$ for each $x \in [0, \pi]$, decreases on its domain. Therefore, it is invertible. The inverse is called the *inverse cosine*, or *arccosine*.

The inverse is denoted by $\arccos(y)$.

The domain of the arccosine is the range of the cosine, that is, $[-1, 1]$.
The range of the arccosine is the domain of the cosine, that is, $[0, \pi]$.

By the definition of inverse function,

$\arccos(\cos x) = x$ for any $x \in [0, \pi]$, and

$\cos(\arccos y) = y$ for any $y \in [-1, 1]$.

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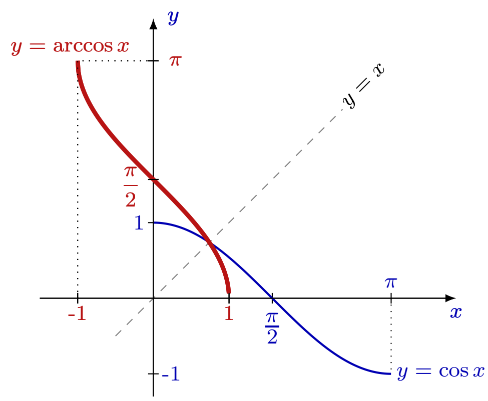
The graph of the inverse cosine

The graphs of a function and its inverse are symmetric about the line $y = x$.

Therefore, the graph of $y = \arccos x$ is obtained from the graph of $y = \cos x$ for $x \in [0, \pi]$, by reflection in the line $y = x$:



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The inverse tangent

The function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, defined by the formula $f(x) = \tan x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, increases on its domain.

Therefore, it is invertible. The inverse is called the *inverse tangent*, or *arctangent*.

The inverse is denoted by $\arctan(y)$.

The domain of the arctangent is the range of the tangent, that is, $\mathbb{R}$.
The range of the arctangent is the domain of the tangent, that is, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

By the definition of inverse function,

$\arctan(\tan x) = x$ for any $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and

$\tan(\arctan y) = y$ for any $y \in \mathbb{R}$.

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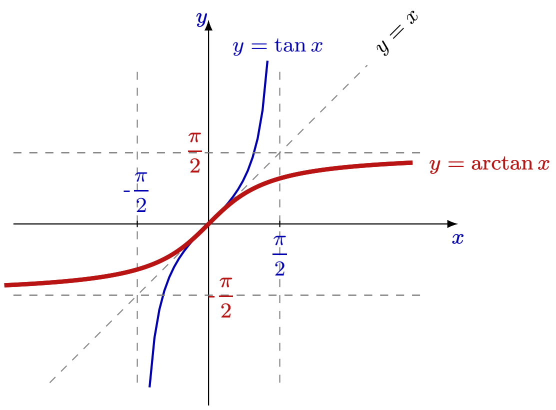
The graph of the inverse tangent

The graphs of a function and its inverse are symmetric about the line $y = x$.
Therefore, the graph of $y = \arctan x$ is obtained from the graph of $y = \tan x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, by reflection in the line $y = x$:



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Warning

In some textbooks, the inverse trigonometric functions

$$y = \arcsin x\,,\ \ y = \arccos x\,,\ \ y = \arctan x\,,$$

are denoted by

$$y = \sin^{-1} x\,,\ \ y = \cos^{-1} x\,,\ \ y = \tan^{-1} x\,.$$

This notation may cause confusion, because it is ambiguous:

for example, $\sin^{-1} x$ may denote either the reciprocal of $\sin x$, that is $\dfrac{1}{\sin x}$,
or the inverse sine, that is $\arcsin x$.

But $\dfrac{1}{\sin x} \neq \arcsin x$.

☞ To avoid possible confusion, use arc-notation for the inverse trigonometric functions.

Summary

In this lecture, we studied

- **logarithms** as inverses of exponentials
- **graphs** of logarithm functions
- **laws** of logarithms
- what the **natural logarithm** $y = \ln x$ is
- the inverse sine $y = \arcsin x$ and its domain, range and graph
- the inverse cosine $y = \arccos x$ and its domain, range and graph
- the inverse tangent $y = \arctan x$ and its domain, range and graph

Comprehension checkpoint

- Why is $e^{\ln x} = x$?
- Is it true that $\ln x^a = a \ln x$?
- What is the domain of the function $y = \ln x$?
- How does the graph of the inverse sine look like?
- Is it true that the domain on the inverse cosine is $[0, \pi]$?
- Is it true that $\arctan x$ is defined for all real x?
- What are the asymptotes of $y = \arctan x$?