Lecture 4

Elementary Functions. Part 2

Objectives

What are elementary functions?

Power, exponential, logarithmic, trigonometric, inverse trigonometric functions

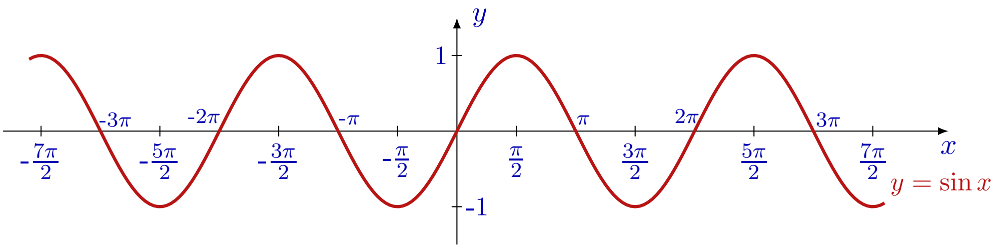
and their sums, differences, products, quotients, and compositions.

In this lecture, we review **trigonometric functions** ($y = \sin x,\ y = \cos x,\ y = \tan x,\ y = \cot x$),

Also, we give the definition of an **inverse function** and discuss which functions have an inverse.

Trigonometric functions: sine

$y = \sin x$



Domain: $\mathbb{R}$

Range: $[-1, 1]$

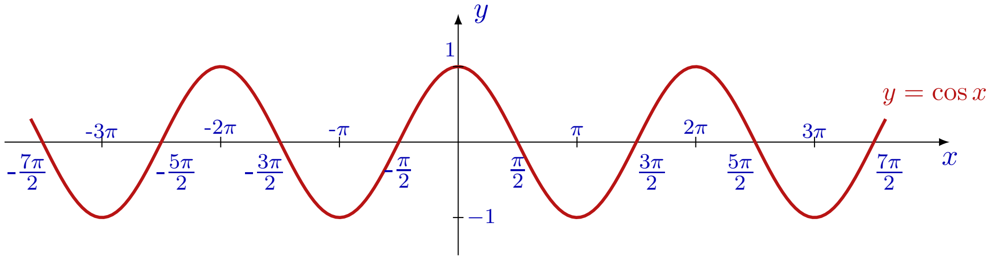
Periodicity: $\sin(x + 2\pi n) = \sin x$ for any integer n and any real x.

Symmetry: $y = \sin x$ is an **odd** function: $\sin(-x) = -\sin x$ for any x.

The graph of $y = \sin x$ is symmetric about the origin.

Trigonometric functions: cosine

$y = \cos x$



Domain: $\mathbb{R}$

Range: $[-1, 1]$

Periodicity: $\cos(x + 2\pi n) = \cos x$ for any integer n and any real x.

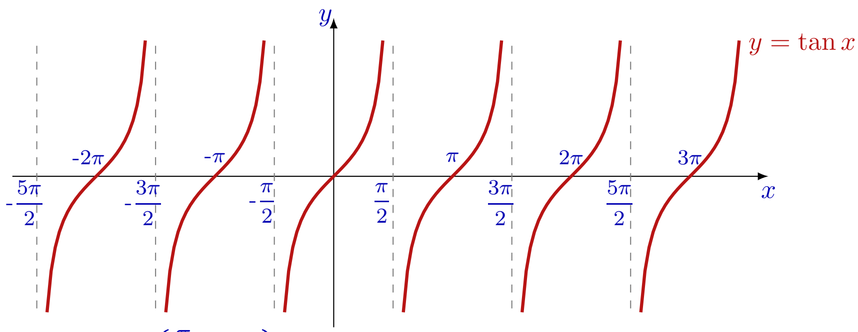
Symmetry: $y = \cos x$ is an **even** function: $\cos(-x) = \cos x$ for any x.

The graph of $y = \cos x$ is symmetric about the y-axis.

Cosine and sine are closely related: $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

Trigonometric functions: tangent

$y = \tan x \quad \left(\tan x = \frac{\sin x}{\cos x}\right)$



Domain: $\mathbb{R} \setminus \left\{\frac{\pi}{2} + \pi n\right\}$, where n is an integer. Range: $(-\infty, \infty)$

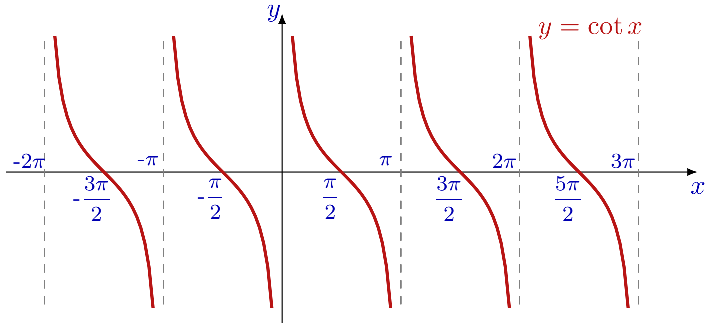
Periodicity: $\tan(x + \pi n) = \tan x$ for any integer n.

Symmetry: $y = \tan x$ is an **odd** function: $\tan(-x) = -\tan x$ for any x in the domain.

The graph of $y = \tan x$ is symmetric about the origin.

Trigonometric functions: cotangent

$y = \cot x \quad \left(\cot x = \dfrac{\cos x}{\sin x}\right)$



Domain: $\mathbb{R} \setminus \{\pi n\}$, where n is an integer. Range: $(-\infty, \infty)$

Periodicity: $\cot(x + \pi n) = \cot x$ for any integer n and any real x.

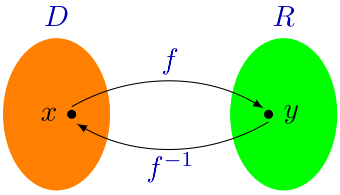
Symmetry: $y = \cot x$ is an **odd** function: $\cot(-x) = -\cot x$ for any x in the domain.

The graph of $y = \cot x$ is symmetric about the origin.

Inverse functions

Definition. A function $f : D \to R$ with domain D and range R is called *invertible* if there exists a function $f^{-1} : R \to D$,
with domain R and range D, which has the following property:
$(f^{-1} \circ f)(x) = x$ for any $x \in D$ and $(f \circ f^{-1})(y) = y$ for any $y \in R$.

The function f^{-1} is called the *inverse* of f.



If $y = f(x)$ is invertible, then

$x = (f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y).$

If f^{-1} is an inverse for f, then $y = f(x) \iff x = f^{-1}(y)$.

Remember: $f^{-1}(f(x)) = x$ for any $x \in D$ and

$f(f^{-1}(y)) = y$ for any $y \in R$.

The inverse function is unique (if it exists).

Warning about notations

⚠ The notation f^{-1} is used in two different meanings.

We just discussed the notion of inverse function. A function inverse for f is denoted by f^{-1}.

Also, f^{-1} denotes the reciprocal of f: $f^{-1} = \frac{1}{f}$.

Do not confuse the notation f^{-1} for the inverse function

and the notation $f^{-1} = \frac{1}{f}$ for the reciprocal.

As a rule, the meaning of f^{-1} is clear from the context.

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Examples of invertible functions

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3$.

f is invertible, and its inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(y) = \sqrt[3]{y}$.

Indeed, $f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x$ for any x, and

$f(f^{-1}(y)) = f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y$ for any y.

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 3x - 2$.
Find the inverse of f.

Solution. Let $y = 3x - 2$. To find the inverse, we have to solve this equation for x in terms of y.

$$y = 3x - 2 \iff y + 2 = 3x \iff \frac{y+2}{3} = x \iff x = \frac{1}{3}y + \frac{2}{3}.$$

Therefore, the inverse of f is $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(y) = \frac{1}{3}y + \frac{2}{3}$.

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The graph of the inverse function

How do we draw the graph of f^{-1}, if we know the graph of f ?

Since $y = f(x) \iff x = f^{-1}(y)$, to draw the graph of f^{-1} in the xy-plane we have to swap the variables in $x = f^{-1}(y)$:

$$x = f^{-1}(y) \quad \text{becomes} \quad y = f^{-1}(x)$$

The swap corresponds to the reflection in the line $y = x$.

Therefore, the graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by the **reflection** in the line $y = x$.

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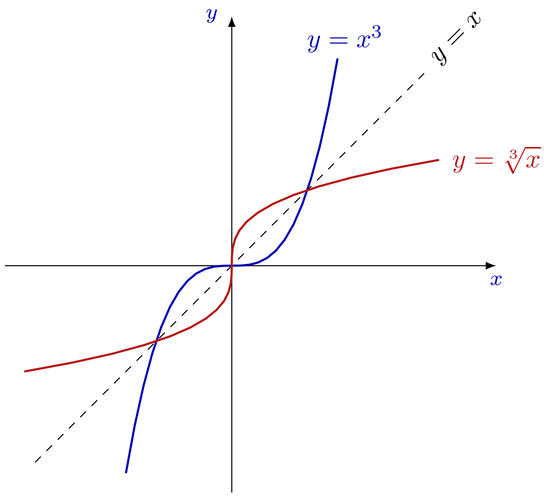
The graph of the inverse function

Example 1. Draw the graphs of the function $f(x) = x^3$ and its inverse in the same coordinate plane.

Solution.



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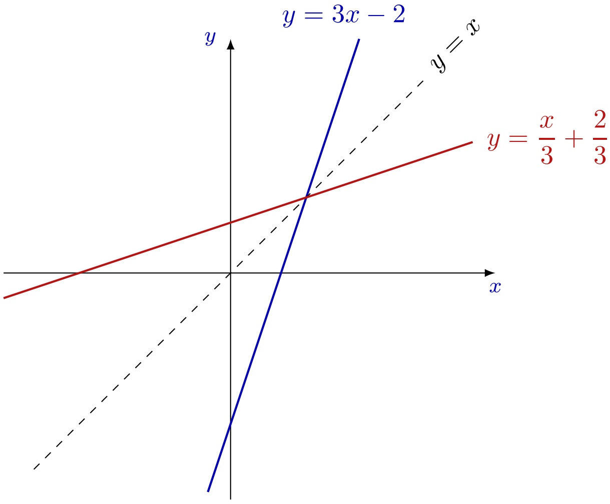
The graph of the inverse function

Example 2. Draw the graphs of the function $f(x) = 3x - 2$ and its inverse in the same coordinate plane.

Solution.

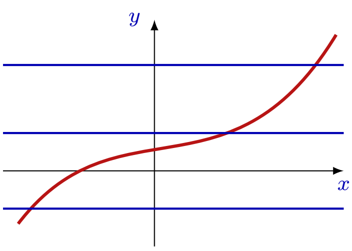


Which functions have an inverse?

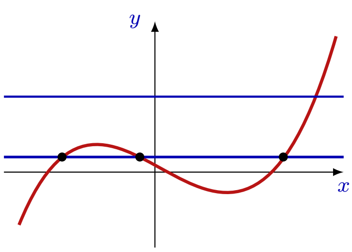
If the function $f : D \to R$ has inverse $f^{-1} : R \to D$, then for any $y \in R$ there exists a **unique** $x \in D$ such that $x = f^{-1}(y)$.

Graphically, this means that **any** horizontal line $y =$ constant intersects the graph of $y = f(x)$ at most once.

This gives the so called *horizontal line test*, used to check if a function is invertible.



this function is invertible



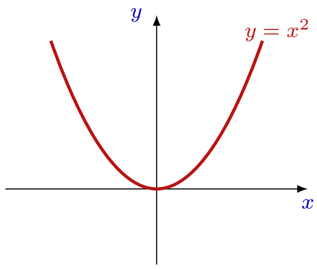
this function is not invertible

Monotonic functions

Definition. A function is called *monotonic* on an interval if
it is either strictly increasing on that interval, or strictly decreasing on that interval.

Example 1. $f(x) = x^2$ is monotonic on $(-\infty, 0]$ (it is strictly decreasing there),
and on $[0, \infty)$ (it is strictly increasing there).

It is **not** monotonic on $(-\infty, \infty)$.



Monotonic functions are invertible

Theorem. If a function is monotonic on an interval, then it is invertible on that interval.

Proof. Assume that f is strictly increasing on an interval I.
(For a strictly decreasing function the reasoning is similar.)

Take any $x_1, x_2 \in I$. If $x_1 \neq x_2$, then $x_1 < x_2$ or $x_1 > x_2$.

In case $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

In case $x_1 > x_2$, we have $f(x_1) > f(x_2)$.

In either case, $f(x_1) \neq f(x_2)$.

This means that for different values of the variable, f takes different values:

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \text{ for all } x_1, x_2 \in I.$$

Let $R = \{f(x) \mid x \in I\}$ be the range of f.
Since f takes different values for different values of the variable,
for each $y \in R$ there exists a **unique** $x \in I$ such that $y = f(x)$.
This means that there exists an inverse function $f^{-1} : R \to I$.

Summary

In this lecture, we have gone over the following:

• **trigonometric** functions

$y = \sin x,\ y = \cos x,\ t = \tan x,\ y = \cot x$ and their domains, ranges and graphs

• the notion of an **inverse** function

• the graphs of a function and its inverse are symmetric about the line $y = x$

• **monotonic** functions are invertible

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Comprehension checkpoint

• Is it true that $\sin(x + \pi) = \sin x$?

• Is it true that $\tan(x + 2\pi) = \tan x$?

• How does the graph of $y = \tan x$ look like?

• Explain why the functions $y = 2x + 1$ and $x = \frac{1}{2}y - \frac{1}{2}$
are inverse to each other.

• Which functions are called monotonic?

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