Lecture 3

# Elementary Functions. Part 1

## Objectives

In the next three lectures, we will discuss the class of **elementary functions**.

This lecture is devoted to

- power functions
- polynomials
- rational functions
- exponential functions.



## Which functions are called elementary

In our calculus course, we are going to deal mostly with *elementary functions*.

They are

power functions ( $x^2,\ \sqrt{x},\ x^{1/3},\ \dots$ ),
exponential functions( $2^x,\ \pi^x,\ e^x,\ \dots$ ),
logarithmic functions ( $\ln x,\ \log_2 x,\ \dots$ ),
trigonometric functions ( $\sin x,\ \cos x,\ \tan x,\ \dots$ ),
inverse trigonometric functions ( $\arcsin x,\ \arctan x,\ \dots$ )

and their sums, differences, products, quotients, and compositions.

For example,

$f(x) = \dfrac{\arcsin\sqrt{x^2-3}}{\ln(x^4+5) - \tan e^{\cos x}}$ is an elementary function.

There are many non-elementary functions, for example

$f(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$ is **not** an elementary function.

## Power functions

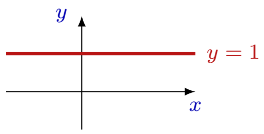
**Definition.** A *power* function is a function of the form $f(x) = x^a$, where $a$ is a given constant.

For example, $y = x,\ y = x^5,\ y = \sqrt[3]{x} = x^{1/3},\ y = x^{2/3}$ are power functions.

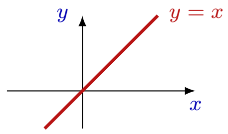
In a power function $f(x) = x^a$, the base $x$ is a variable, and the exponent $a$ is a constant.

The appearance of the graph of a power function depends on the constant $a$.

For $a = 0$, $y = x^0 = 1$




For $a = 1$, $y = x^1 = x$

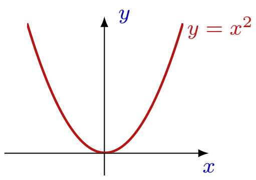


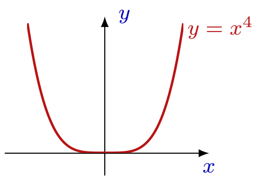


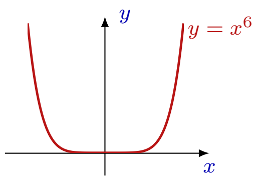

## Power functions $y = x^n$

If $n$ is an integer greater than $1$, then the overall shape of the graph of $y = x^n$ is determined by the parity of $n$ (whether $n$ is even or odd).

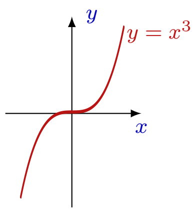
If $n$ is even, then the graph has a shape similar to the parabola $y = x^2$:

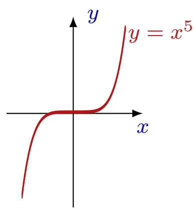


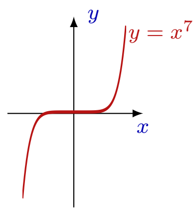




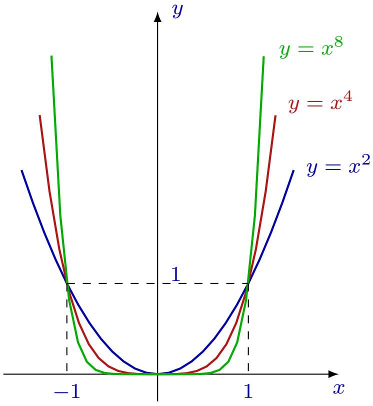

If $n$ is odd, then the graph has a shape similar to the cubic parabola $y = x^3$:

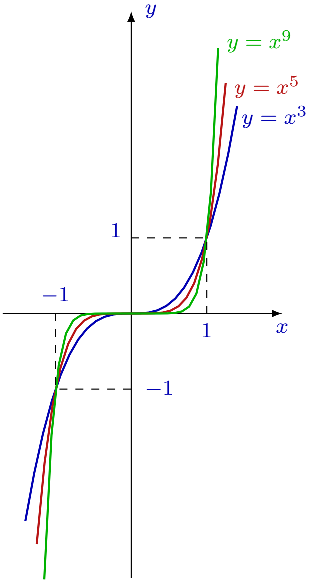






## All together

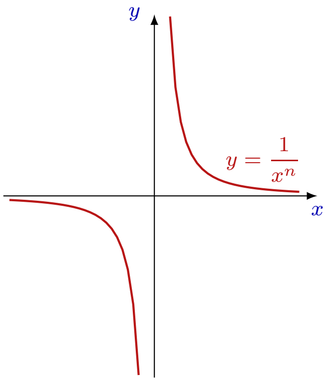


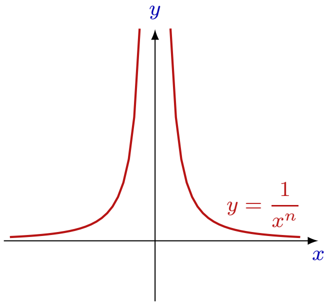



## Power functions $y = \frac{1}{x^n} = x^{-n}$

If $n$ is a positive integer greater than $1$, then the overall shape of the graph of $y = \dfrac{1}{x^n}$ is determined by the parity of $n$ (whether $n$ is even or odd).



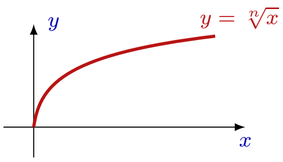

$n$ is odd



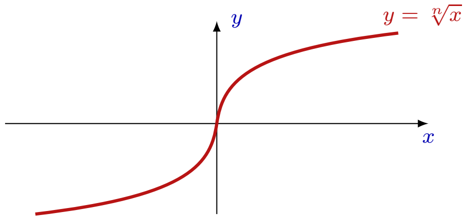

$n$ is even

## Power functions $y = \sqrt[n]{x} = x^{1/n}$

If $n$ is an positive integer, then the overall shape of the graph of $y = \sqrt[n]{x} = x^{1/n}$ is determined by the parity of $n$ (whether $n$ is even or odd).




$n$ is even

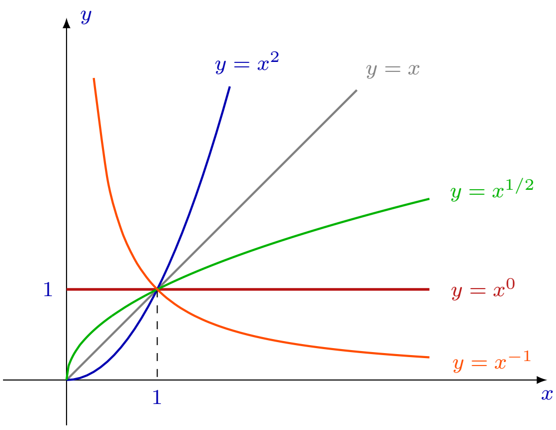



$n$ is odd



## $y = x^n$ all together

Here are the graphs of $y = x^n$ for some rational $n$ and $x > 0$:



## Polynomials

**Definition.** Let $n$ be a non-negative integer.
A *polynomial* in the variable $x$ of degree $n$ is a function of the form

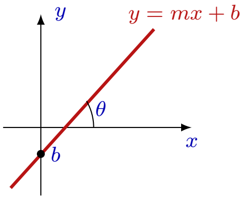
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\,,$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants (they are called the *coefficients* of the polynomial) and $a_n \neq 0$. The integer $n$ is called the *degree* of the polynomial.

A polynomial is the sum of *monomials* $a_k x^k$.

**Example 1.** A *linear function* $y = mx + b$, where $m$ and $b$ are constants, is a polynomial of degree $1$.

The graph of a linear function is a straight line:



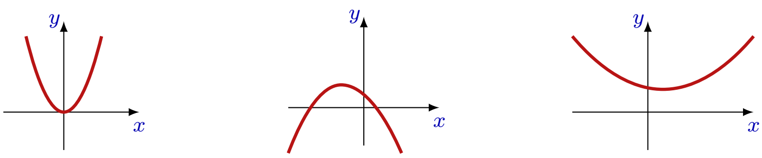

$m$ is the *slope* of the line, $m = \tan\theta$
$b$ is the $y$-intercept.



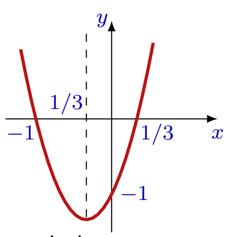
## Quadratic functions

**Example 2.** A *quadratic function* $y = ax^2 + bx + c$, where $a, b, c$ are constants, is a polynomial of degree $2$.

The graph of a quadratic function is a *parabola*. Its appearance depends on the coefficients $a, b, c$.




☞ Check your calculus readiness:
Draw the graph of $y = 3x^2 + 2x - 1$.
How long did it take?
If less than 2 minutes, then you are ready!




If you are not quite ready, let us review how to draw a parabola.

## How to draw a parabola

To draw a parabola $y = ax^2 + bx + c$, follow the following scheme.

1. Determine the parabolas **vertex**. It is located at the point $\left(\frac{-b}{2a}, y\left(\frac{-b}{2a}\right)\right)$

2. Draw the **axis of symmetry**. It is the vertical line $x = \frac{-b}{2a}$.

3. Determine if the parabola opens **up** ( $a > 0$ ) or **down** ( $a < 0$ )

4. Determine the intercepts. The $y$-intercept is $(0, c)$,
the $x$-intercepts are $(x_1, 0)$, $(x_2, 0)$, where $x_1, x_2$ are the solutions of the quadratic equation $ax^2 + bx + c = 0$ (if there are any).

5. Plot the vertex, the axis of symmetry, the intercepts on a coordinate system and construct a smooth parabola.

**Exercise.** Use the scheme above to draw the parabola $y = 3x^2 + 2x - 1$.



## Rational functions

**Definition.** A *rational function* is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}, \text{ where } P(x) \text{ and } Q(x) \text{ are polynomials.}$$

For example,

$$f(x) = \frac{1}{4x^2 + 3x - 1} \quad \text{and} \quad f(x) = \frac{2x^3 - x + 1}{x^2 - 3} \quad \text{are rational function.}$$

The domain of a rational function is the set of all real numbers for which the denominator is not zero.

## Exponential functions

**Definition.** An *exponential* function is a function of the form $f(x) = a^x$,
where $a$ is a given positive constant.

For example, $y = 2^x$, $y = \left(\frac{3}{5}\right)^x$, $y = e^x$, $y = (\sqrt{2})^x$ are exponential functions.

In an exponential function $f(x) = a^x$, the base $a$ is a constant, and the exponent $x$ is a variable.

⚠ **Warning.** Don't confuse exponential ($y = a^x$) and power ($y = x^a$) functions!

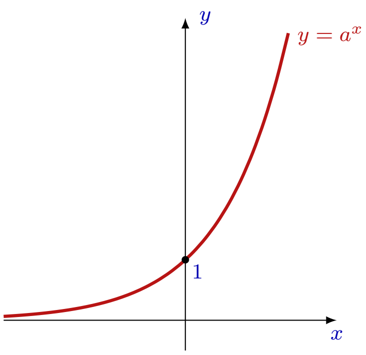
The appearance of the **graph** of an exponential function depends on $a$.

The graph of each exponential function $f(x) = a^x$
intersects the $y$-axis at the point $(0, 1)$, since $f(0) = a^0 = 1$.

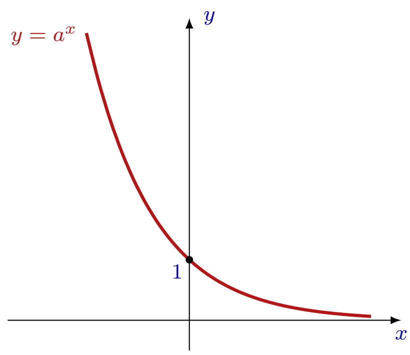
If $a = 1$, then the exponential function is $y = 1^x = 1$. It is a constant function.

14 / 21

## The graphs of exponential functions




$a > 1$



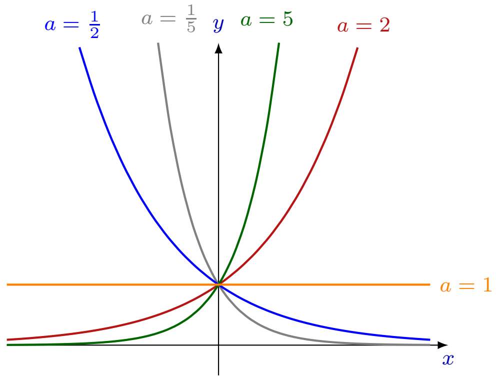

$0 < a < 1$

**Remember:** The domain of any exponential function is $\mathbb{R}$, since $y = a^x$ is defined for **all** $x$.

The range of $y = a^x$ is $(0, \infty)$, since $a$ is positive and therefore $a^x > 0$ for any $x$.

15 / 21

## Exponential functions $y = a^x$ all together




16 / 21

## The laws of exponents (reminder)

Let $a > 0$, $b > 0$, and $x, y$ be any real numbers. Then

$$a^0 = 1 \qquad a^{x+y} = a^x a^y$$

$$a^{-x} = \frac{1}{a^x} \qquad a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

17 / 21

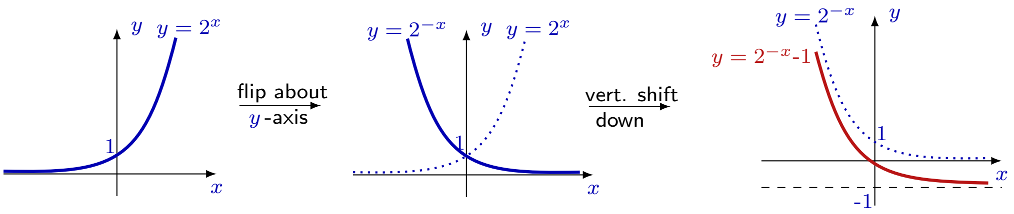
## Graphing exponential functions

**Problem.** Draw the graph of the function $f(x) = 2^{-x} - 1$ .

**Solution.** We need to know the graph of the standard function $y = 2^x$ ,
and to be familiar with graph transformations.

Steps to follow in drawing the graph:

step 1 $y = 2^x$ ⟶ step 2 $y = 2^{-x}$ ⟶ step 3 $y = 2^{-x} - 1$



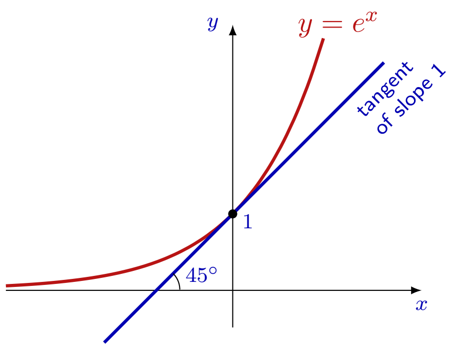



## The number $e$

This is the most amazing number in mathematics!

It may be defined in many different ways. For example, like this:

The number $e$ is the base of the exponential function
whose graph has the tangent line at $(0, 1)$ with slope $1$ .




The exact definition of the *tangent line* to the graph of a function at a point will be given later.

## Summary

In this lecture we reviewed the following topics:

- **power** functions $f(x) = x^a$ and their graphs
- **polynomials**
- **quadratic** functions and their graphs
- **exponential** functions $f(x) = a^x$ and their graphs
- **Laws** of exponents
- the number $e$.



## Comprehension checkpoint

- What are the names of the following functions:

$y = \sqrt[3]{x^2}$

$y = \left(\frac{2}{3}\right)^x$

$y = -5x^3 + 6x^2 + 1$

$y = \frac{3x+2}{-x^2+4x+2}$

- What is $e$?
- How does the graph of $y = e^x$ look like?