Lecture 2

Operations on Functions

Objectives 2
Operations on functions 3
Composition of functions 4
Composition of more than two functions 5
Graph transformations 6
Vertical shifts 7
Graph transformations: horizontal shifts 8
Graph transformations: vertical stretch/shrink 9
Graph transformations: horizontal stretch/shrink 10
Graph transformations: reflections about the axes. 11
Using graph transformation for graphing 12
Using graph transformation for graphing 13
Summary 14
Comprehension checkpoint 15

Objectives

In this lecture, we will consider **operations** on functions:

- addition
- subtraction
- multiplication
- division
- **composition**

We will go over **transformations** of graphs:

- vertical and horizontal shifts
- vertical and horizontal stretch/shrink
- reflections about the coordinate axes

Operations on functions

Given two functions f and g, one can construct new functions by taking their sum, difference, product, or quotient.

The sum $f+g$, difference $f-g$, product fg, and quotient f/g of two functions f and g are defined as follows:

$$(f+g)(x) = f(x)+g(x), \qquad (f-g)(x) = f(x)-g(x),$$

$$(fg)(x) = f(x)g(x), \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The domain of $f+g$, $f-g$, fg is the **intersections** of the domains of f, g.
To obtain the domain of f/g, one needs to intersect the domains of f and g, and delete the points where $g=0$.

Example. Find the domain of $f(x) = \dfrac{\sqrt{x}}{x-1}$.

Solution. The function f is the quotient of two functions, $f_1(x)=\sqrt{x}$, $f_2(x)=x-1$.
f_1 is defined for all $x \geq 0$, f_2 is defined for all x.
The denominator of the fraction should be different from 0, so $x \neq 1$.
Therefore, the domain of f is $[0,1) \cup (1,\infty)$.

Composition of functions

Definition. The *composition* of the functions f and g is a function $g \circ f$ defined by

$$(g \circ f)(x) = g(f(x))\,.$$

f and g are called the *inner* and *outer* functions of the composition.
In the composition $g \circ f$, f is performed first, and g is performed second:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & f(x) \arrow[r, "g"] & g(f(x)) \end{tikzcd}$$

Example. Let $f(x) = |x|$, $g(x) = x - 1$. Find $g \circ f$ and $f \circ g$.

Solution. $(g \circ f)(x) = g(\underbrace{f(x)}_{|x|}) = g(|x|) = |x| - 1\,.$

$(f \circ g)(x) = f(\underbrace{g(x)}_{x-1}) = f(x-1) = |x - 1|\,.$ ⚠ $g \circ f \neq f \circ g$

Composition of more than two functions

The composition of more than two functions is defined similarly.

For example, the composition of f_1, f_2 and f_3 is defined as follows:

$(f_3 \circ f_2 \circ f_1)(x) = f_3(f_2(f_1(x)))\,.$

Example. The function $f(x) = \cos^2(x+5)$ is a composition of three functions:

$f_1(x) = x + 5$, $f_2(x) = \cos x$, $f_3(x) = x^2$:

$$\begin{tikzcd} x \arrow[r, "f_1"] \arrow[rrr, bend right, "f_3 \circ f_2 \circ f_1"'] & x+5 \arrow[r, "f_2"] & \cos(x+5) \arrow[r, "f_3"] & \cos^2(x+5) \end{tikzcd}$$

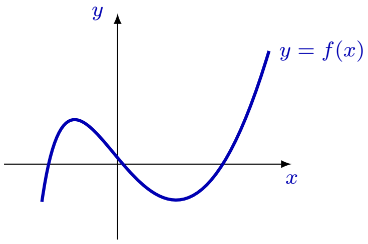
Extra questions: What is $f_1 \circ f_2 \circ f_3$? $(f_1 \circ f_2 \circ f_3)(x) = \cos(x^2) + 5\,.$

What is $f_3 \circ f_1 \circ f_2$? $(f_3 \circ f_1 \circ f_2)(x) = (\cos x + 5)^2\,.$

Graph transformations

We review the transformations of the graphs by solving the following problem.

Problem. Given the graph of a function $y = f(x)$.



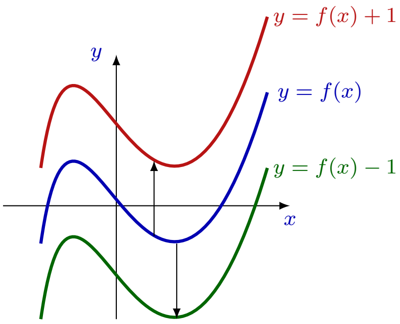
Construct the graphs of the following functions:

$y = f(x) + 1\,,\ y = f(x) - 1\,,\ y = f(x+1)\,,\ y = f(x-1)\,,$

$y = 2f(x)\,,\ y = \frac{1}{2}f(x)\,,\ y = -f(x)\,,\ y = f(-x)\,,\ y = f(2x)\,,\ y = f\left(\frac{x}{2}\right).$

Vertical shifts

Solution.



$$f(x) + 1$$

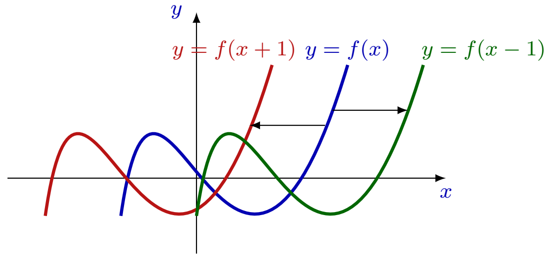
$\big\uparrow$ vertical shift by 1 unit up

$$f(x)$$

$\big\downarrow$ vertical shift by 1 unit down

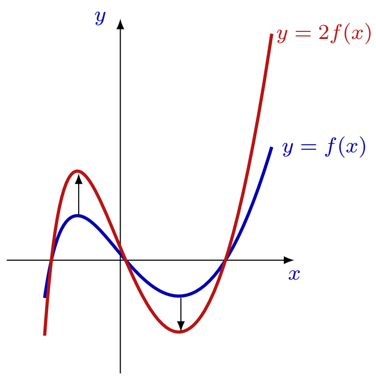
$$f(x) - 1$$

Graph transformations: horizontal shifts

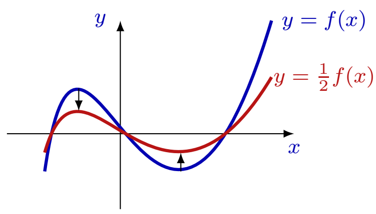


$f(x+1) \xleftarrow{\text{horizontal shift by } 1 \text{ unit to the left}} f(x) \xrightarrow{\text{horizontal shift by } 1 \text{ unit to the right}} f(x-1)$

Graph transformations: vertical stretch/shrink

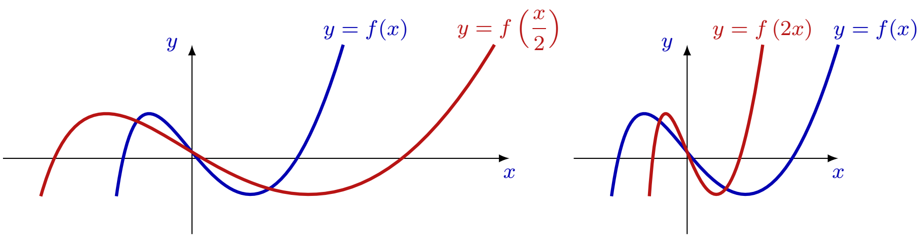


vertical stretch by the factor of 2
away from the x-axis



vertical shrink by the factor of 2
towards the x-axis

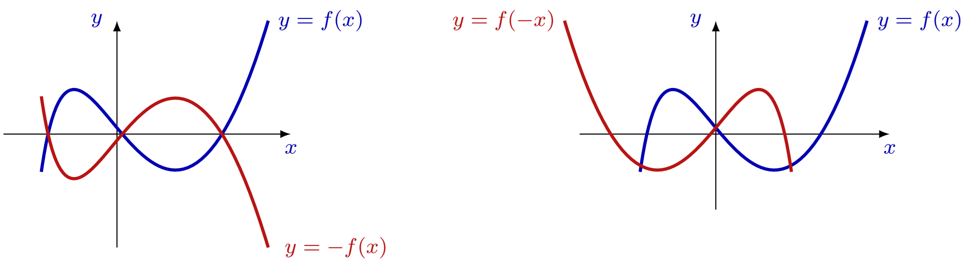
Graph transformations: horizontal stretch/shrink



horizontal stretch by the factor of 2
away from the y-axis

horizontal shrink by the factor of 2
towards the y-axis

Graph transformations: reflections about the axes



reflection about the x-axis

reflection about the y-axis

Using graph transformation for graphing

Example. Draw the graph of the function $y = \dfrac{x-1}{x+1}$.

Solution. We use polynomial division to break the quotient into the sum of two simpler functions:

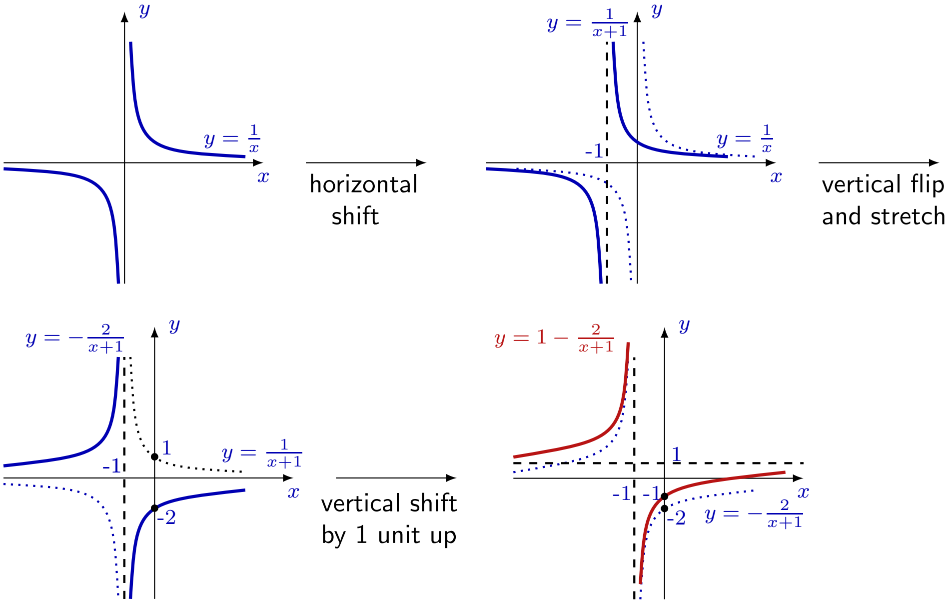
$$y = \frac{x-1}{x+1} = \frac{(x+1)-2}{x+1} = 1 - \frac{2}{x+1}.$$

Now we organize our drawing into several steps:

step 1		step 2		step 3		step 4
$y = \frac{1}{x}$	$\longrightarrow$	$y = \frac{1}{x+1}$	$\longrightarrow$	$y = -\frac{2}{x+1}$	$\longrightarrow$	$y = 1 - \frac{2}{x+1}$

We draw, one after the other, the graphs of the four functions above.

Using graph transformation for graphing



Summary

In this lecture we studied the following topics:

- how to add, subtract, multiply and divide functions
- what a **composition** of functions is
- how to construct graphs of functions using **graphs transformations**

Comprehension checkpoint

- If $f(x) = x^2 - 1$ and $g(x) = \tan x$, what are $g \circ f$ and $f \circ g$?

► $(g \circ f)(x) = \tan(x^2 - 1)$, $(f \circ g)(x) = \tan^2 x - 1$.

- Given the graph of $y = x^2$, construct the graph of $y = -(x-1)^2 + 2$

►