Lecture 1

General Information about Functions

Objectives

In our first lecture, we discuss

- the definition of a **function**
- the **domain** and **range** of a function
- **piecewise-defined** functions
- **even** and **odd** functions
- **increasing** and **decreasing** functions.

What is a function?

Loosely speaking, a function expresses the **dependence** of a quantity on another quantity.

The precise meaning of a function is given in the following definition:

Definition. Let D and C be sets of real numbers: $D \subset \mathbb{R}$, $C \subset \mathbb{R}$.
A *function* f from D to C is a **rule**
that assigns to each element in D exactly one element in C.

Notation. $f : D \longrightarrow C$
$x \mapsto f(x)$ or $y = f(x)$.

$$f : \underbrace{D}_{\substack{\textit{domain} \\ \text{of } f}} \longrightarrow \underbrace{C}_{\substack{\textit{codomain} \\ \text{of } f}} \qquad \underbrace{y}_{\substack{\textit{dependent} \\ \text{variable}}} = f(\underbrace{x}_{\substack{\textit{independent} \\ \text{variable}}})$$

Domain and range

For each number x in the domain $D,$ the function f returns the number $f(x)$ in the codomain $C.$ This number is called the *value* of f at $x.$

The set of all values of a function f is called the *range* of $f.$
The range is a subset of the codomain.

The *graph* of a function f is the set of ordered pairs $\{(x, f(x)) \mid x \in D\}.$
The graph of a function is a subset of $\mathbb{R}^2.$

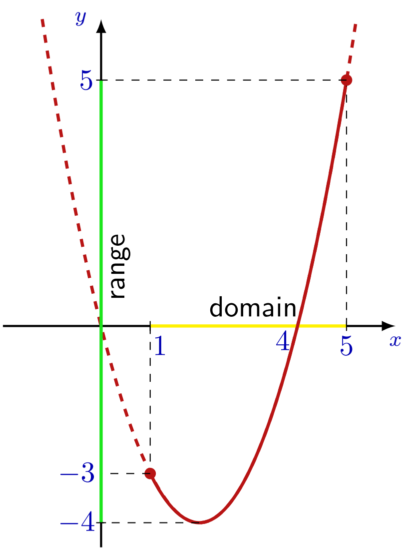
Example. Let $f : [1,5] \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2 - 4x.$
Determine the domain, codomain and range of $f.$ Draw the graph of $f.$

Solution. The domain is the interval $[1,5],$ the codomain is $\mathbb{R}$.

The graph of $f : [1,5] \longrightarrow \mathbb{R}$ is the set of all points $\{(x,y)\}$ in the plane
such that $x \in [1,5]$ and $y = x^2 - 4x.$
That is, the graph of f is a part of the parabola $y = x^2 - 4x,$ where x takes values in $[1,5]$.

Graph of a function

To draw the graph of $y = f(x),$ we plot the parabola $y = x^2 - 4x$:



Restrict x to $[1,5],$
and cut off
the corresponding piece of the parabola.
The range is $[-4,5].$

Remember:
the domain appears on the x-axis,
the range appears on the y-axis.

How to find the domain

Often the function is given by a single formula,

like $y = x^2 - 4x$, or $f(x) = \dfrac{x+1}{(x+2)(x-3)}$ or $f(x) = \sqrt{x}$.

In such cases the domain is assumed to be the **maximal** set of x-values for which the formula makes sense.

For example, for the function $y = x^2 - 4x$, the domain is $\mathbb{R}$, since the expression $x^2 - 4x$ is defined for **all** values of x.

For the function $f(x) = \dfrac{x+1}{(x+2)(x-3)}$, the domain is

$$\mathbb{R} \setminus \{-2, 3\} = (-\infty, -2) \cup (-2, 3) \cup (3, \infty),$$

since the formula makes sense for all x besides -2 and 3.

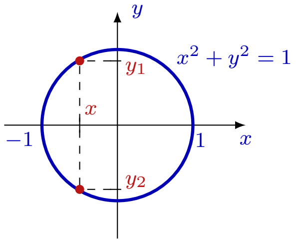
The domain of the function $f(x) = \sqrt{x}$ is $[0, \infty)$,
since the $\sqrt{x}$ makes sense for all non-negative values of x,
and is not defined for negative values of x.

6 / 20

Warning

⚠ **Warning.** Not every curve in the plane is the graph of a function.

For example, the circle $\{(x, y) \mid x^2 + y^2 = 1\}$ is **not** the graph of a function. Why not?



A function f has **only one** value $f(x)$ for each x in the domain.
On the circle, for each $x \in (-1, 1)$
there are **two** values of y, namely y_1 and y_2, for which $x^2 + y^2 = 1$.
Therefore, the circle is not the graph of a function.

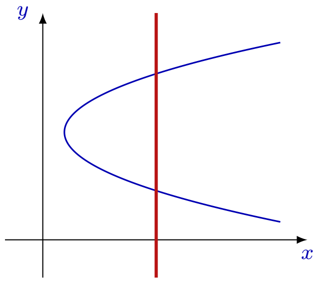
It is correct to say that the circle is the graph of the **equation** $x^2 + y^2 = 1$.

7 / 20

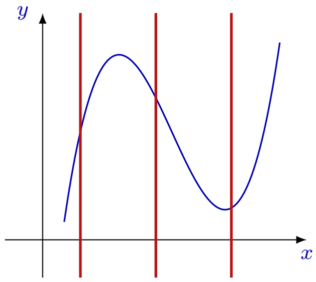
Vertical line test

A curve on a plane is the graph of a function if and only if

no vertical line intersects the curve more than once.



not the graph of a function



the graph of a function

Piecewise defined functions

Some functions are defined by multiple formulas on different parts of their domain.

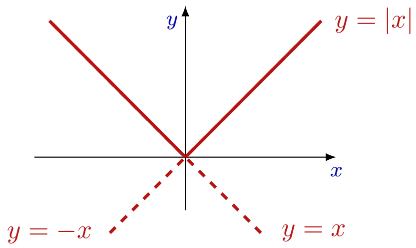
They are called *piecewise defined* functions.

Example 1 (absolute value function).

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$|0| = 0$, $|2| = 2$, $|-3| = 3$. Remember: $|x| \geq 0$ for any x.

The domain of $y = |x|$ is $\mathbb{R}$, the range is $[0, \infty)$.



Piecewise defined functions

Example 2. Draw the graph of the function

$$f(x) = \begin{cases} -x, & x \leq -1 \\ 0, & -1 < x \leq 2 \\ x^2 - 4, & x > 2 \end{cases}$$

Solution. The domain $\mathbb{R}$ splits into three parts:

$$\mathbb{R} = (-\infty, -1] \cup (-1, 2] \cup (2, \infty)\,.$$

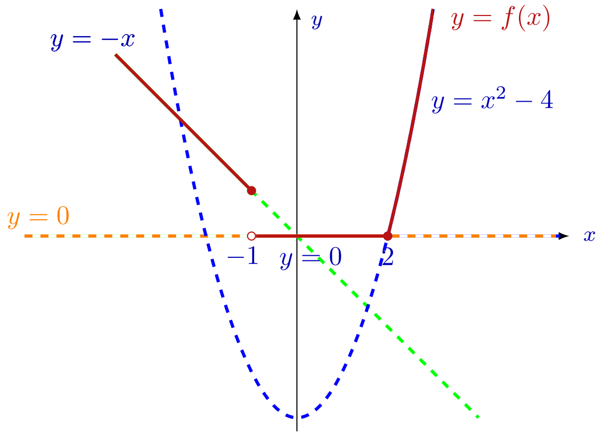
On each part, f is defined by its own formula:

x	$(-\infty, -1]$	$(-1, 2]$	$(2, \infty)$
$f(x)$	$-x$	0	$x^2 - 4$

Piecewise defined functions

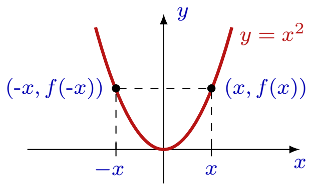
x	$(-\infty, -1]$	$(-1, 2]$	$(2, \infty)$
$f(x)$	$-x$	0	$x^2 - 4$

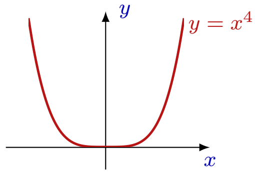
We construct the graph of f piece by piece.

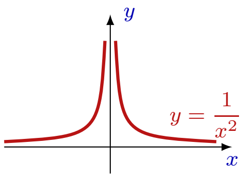


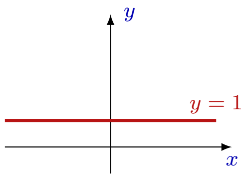
Even functions

Definition. A function f is called *even* if
its domain is symmetric about the origin (that is, for each $x \in D$, we have $-x \in D$)
and $f(-x) = f(x)$ for all x in the domain.
The graph of an even function is **symmetric** about the y-axis.
Examples of even functions: $y = x^2$, $y = x^4$, $y = x^6$.
In general, $f(x) = x^{2n}$ is an even function for each integer n. Indeed,
$f(-x) = (-x)^{2n} = ((-x)^2)^n = (x^2)^n = x^{2n} = f(x)$.



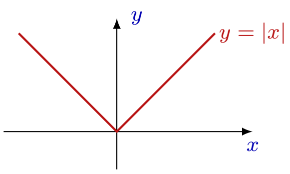


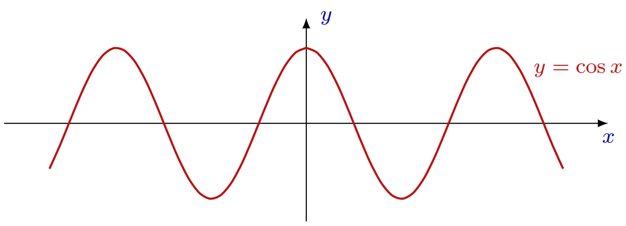




Even functions

More examples of even functions: $y = |x|$, $y = \cos x$.





Even functions

The sum, difference, product and quotient of even functions is an even function.

Example. Prove that $f(x) = 3x^8 - x^2\cos(5x)$ is an even function.

Solution. $f(x)$ is defined for **all** x. To prove that f is even,
we have to show that $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

Take any real number x. Then

$$f(-x) = 3(-x)^8 - (-x)^2\cos(5(-x)) = 3x^8 - x^2\cos(-5x)$$
$$= 3x^8 - x^2\cos(5x) = f(x).$$

Keep in mind that $y = \cos(5x)$ is an even function, that is $\cos(-5x) = \cos(5x)$.

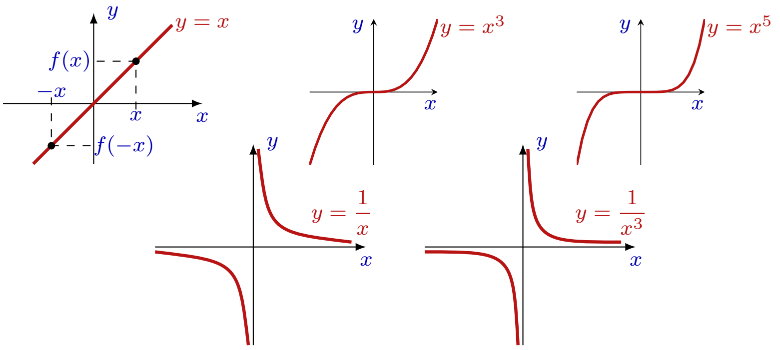
Therefore, $f(-x) = f(x)$ for all real x, and therefore,

$f(x) = 3x^8 - x^2\cos(5x)$ is an even function.

Remark. $f(x)$ is obtained from **even** functions $y = 3x^8$, $y = x^2$, $y = \cos(5x)$
by operations of multiplication and subtraction. That's why $f(x)$ is even.

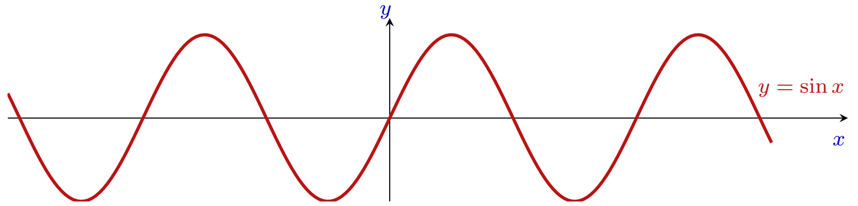
Odd functions

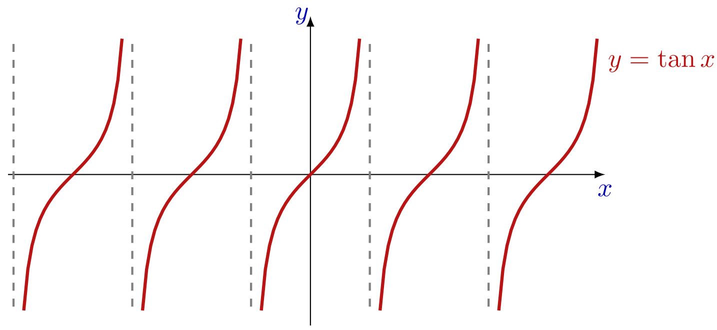
Definition. A function f is called *odd* if
its domain is symmetric about the origin (that is, for each $x \in D$ we have $-x \in D$)
and $f(-x) = -f(x)$ for all x in the domain.
The graph of an odd function is **symmetric** about the origin.
Examples of odd functions: $y = x$, $y = x^3$, $y = x^5$.
In general, $f(x) = x^{2n+1}$ is an odd function for each integer n. Indeed,
$f(-x) = (-x)^{2n+1} = (-x)^{2n}(-x) = x^{2n}(-x) = -x^{2n+1} = -f(x)$.



Odd functions

More examples of odd functions: $y = \sin x$, $y = \tan x$.



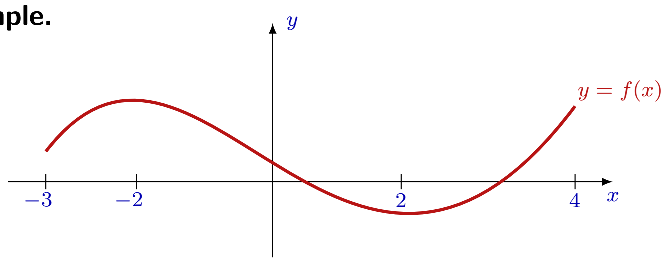


Increasing and decreasing functions

Definition. A function f is called (strictly) *increasing* on an interval I,
if for any $x_1, x_2 \in I$, such that $x_1 < x_2$, one has $f(x_1) < f(x_2)$.

A function f is called (strictly) *decreasing* on an interval I,
if for any $x_1, x_2 \in I$, such that $x_1 < x_2$, one has $f(x_1) > f(x_2)$.

Example.



The function f is increasing on $[-3, -2]$, decreasing on $[-2, 2]$, and increasing on $[2, 4]$.

Control question: Is f increasing on $[-3, -2] \cup [2, 4]$? (Spoiler: No!)

Increasing and decreasing functions

Example. Prove that the function $f(x) = x^2 - 2x$ decreases on $(-\infty, 1]$.

Solution. To prove that f is decreasing on $(-\infty, 1]$, we have to show that if $x_1 < x_2$, then $f(x_1) > f(x_2)$ for any $x_1, x_2 \in (-\infty, 1]$.

Indeed, for any $x_1 < x_2$ in the interval $(-\infty, 1]$,

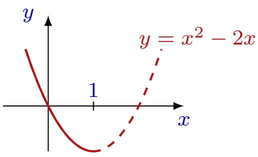
$$f(x_1) - f(x_2) = (x_1^2 - 2x_1) - (x_2^2 - 2x_2) = (x_1^2 - x_2^2) - 2(x_1 - x_2)$$
$$= (x_1 - x_2)(x_1 + x_2) - 2(x_1 - x_2) = (x_1 - x_2)(x_1 + x_2 - 2)\,.$$

Since $x_1 < x_2$, we have $x_1 - x_2 < 0$.

Since $x_1 < x_2 \le 1$, we have $x_1 + x_2 < 2$ and, therefore, $x_1 + x_2 - 2 < 0$.

So $f(x_1) - f(x_2) = \underbrace{(x_1 - x_2)}_{<0}\underbrace{(x_1 + x_2 - 2)}_{<0} > 0$, and $f(x_1) > f(x_2)$.

Hence for any $x_1, x_2 \in (-\infty, 1]$,
$x_1 < x_2 \implies f(x_1) > f(x_2)$,
therefore, f is decreasing on $(-\infty, 1]$



Summary

In this lecture, we studied the following topics:

- the definition of a **function**, its **domain** and **range**
- that is the **graph** of a function and how to determine if a curve is the graph of a function
- what **even** and **odd** functions areand how to prove whether a function is even or odd
- what it means that a function is **increasing** or **decreasing** on an interval and how to check this.

Comprehension checkpoint

We conclude the lecture with a few questions aimed to check how you mastered the material.

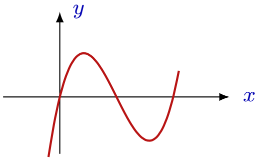
• What is the domain of the function $f(x) = \dfrac{\sqrt{x-1}}{x-2}$?

► $[1, 2) \cup (2, \infty)$

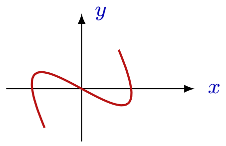
• The graph of the function $y = 3x^4 + \dfrac{1}{x^2}$ is symmetric about the y-axis.
Why is it so?

► The function $y = 3x^4 + \dfrac{1}{x^2}$ is **even** as the sum of two even functions.
Therefore, its graph is symmetric about the y-axis.

• Is the curves below are graphs of functions?



► Yes



► No