Homework

1. Let a, b, c, d be real numbers. Compute the product $(a+bi)(c+di)$ and rewrite it in the form $x+yi$.
2. Rewrite the number $\sqrt{3}+i$ in the polar form $re^{i\theta}$.
3. Rewrite the number $(-5+5i)^3$ in polar form.
4. Write the number $8e^{5\pi/6}$ in the form $a+bi$.
5. Solve for the general solution to the differential equation $y'' - 6y' + 5 = 0$.
6. Solve for the general solution to the differential equation $y'' - 6y' + 9y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = -2$, $y'(0) = 1$.
7. Solve for the general solution to the differential equation $y'' - y' + y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = 2$, $y'(0) = 0$.

Solutions

1. Let a, b, c, d be real numbers. Compute the product $(a+bi)(c+di)$ and rewrite it in the form $x+yi$ where x, y are real numbers expressed in terms of a, b, c, d.

   Solution:

$$\begin{aligned}(a+bi)(c+di) &= (ac+adi+bci+bdi^2)\\ &= (ac+adi+bci-bd)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

2. Rewrite the number $\sqrt{3}+i$ in the polar form $re^{i\theta}$.

   Solution: $\theta = \arctan(1/\sqrt{3}) = \pi/6$ or $\pi/6+\pi$ since the range of arctan only includes angles in quadrants 1 and 4. Seeing that $\sqrt{3}+i$ lies in quadrant 1, $\theta = \pi/6$. We now need to find r. $r = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$. Hence, the polar form of our number is $2e^{i\pi/6}$.

3. Rewrite the number $(-5+5i)^3$ in polar form.

   Solution: Since it is easier to compute powers of numbers when they are in polar form, we start by converting $-5+5i$ into polar form. We first pull out the common factor of 5 to get $5(-1+i)$. Since this point lies in quadrant 2 and the real and imaginary parts have equal length, we have that $\theta = 3\pi/2$. The length of $-1+i$ is $\sqrt{1^2+1^2} = \sqrt{2}$. Hence, $-5+5i = 5(-1+i) = 5(\sqrt{2}e^{i3\pi/2}) = 5\sqrt{2}e^{i3\pi/2}$. Hence,

$$\begin{aligned}(-5+5i)^3 = (5\sqrt{2}e^{i3\pi/2})^3 &= (5\sqrt{2})^3 e^{i3\cdot 3\pi/2}\\ &= 125\cdot 2\sqrt{2}e^{i9\pi/2} = 250\sqrt{2}e^{i\pi/2}.\end{aligned}$$

4. Write the number $8e^{5\pi/6}$ in the form $a+bi$.

   Solution:

$$\begin{aligned}8e^{5\pi/6} &= 8\cos(5\pi/6)+8i\sin(5\pi/6) = 8(-\sqrt{3}/2)+8i(1/2)\\ &= -4\sqrt{3}+4i\end{aligned}$$

5. Solve for the general solution to the differential equation $y''-6y'+5=0$.

   Solution: Since this is a second order linear differential equation with constant coefficients, we may solve it by using the characteristic equation: $r^2-6r+5=0$. The general solution to this equation has the form $y = C_1e^{r_1x}+C_2e^{r_2x}$ where C_1, C_2 are arbitrary constants and r_1, r_2 are roots of the characteristic equation, satisfying $r_1 \neq r_2$. Factoring the characteristic equation, we get $(r-5)(r-1)=0$. Hence, the roots are $r_1 = 5$, $r_2 = 1$. Hence, the general solution is the function $y = C_1e^{5x}+C_2e^{x}$.

6. Solve for the general solution to the differential equation $y'' - 6y' + 9y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = -2$, $y'(0) = 1$.

   Solution: The characteristic equation is $r^2 - 6r + 9 = 0$. Factoring, we get $(r-3)(r-3) = 0$. Since both roots are at $r = 3$, the general solution is $y = C_1 e^{3x} + C_2 x e^{3x}$. Plugging in the initial condition $y(0) = -2$, we have $-2 = C_1 e^0 + C_2(0)e^0 = C_1$. Hence, $C_1 = -2$. To utilize the other initial condition, $y'(0) = 0$, we must first compute y'. $y' = 3C_1 e^{3x} + (C_2 e^{3x} + 3C_2 x e^{3x})$. Plugging in the point $(0, 1)$ and $C_1 = -2$, we have

   $$1 = 3(-2)e^0 + C_2 e^0 + 3C_2(0)e^0 \qquad = -6 + C_2$$

   Hence, $C_2 = 7$. So the particular solution is $y = -2e^{3x} + 7xe^{3x}$.

7. Solve for the general solution to the differential equation $y'' - y' + y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = 2$, $y'(0) = 0$.

   Solution: The characteristic equation is $r^2 - r + 1 = 0$. Using the quadratic formula to find the roots, we have $r = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2}$. Since these are complex roots and we only care about a real solution, the general solution is of the form $y = e^{x/2}(A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x))$.

   Utilizing the initial conditions, we solve for the particular solution. Since $y(0) = 2$, we have $2 = e^0(A\cos(0) + B\sin(0)) = A$. So $A = 2$. To find B, we must first compute y'. Using product rule, we have

   $$y' = \frac{1}{2}e^{x/2}(A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x)) + e^{x/2}(-\sqrt{3}A\sin(\sqrt{3}x) + \sqrt{3}B\cos(\sqrt{3}x))$$

   Plugging in the initial condition $y'(0) = 0$ gives us

   $$0 = \frac{1}{2}e^0(A\cos(0) + B\sin(0)) + e^0(-\sqrt{3}A\sin(0) + \sqrt{3}B\cos(0)) = \frac{1}{2}(A) + 1(\sqrt{3}B).$$

   Hence, $B = (-\frac{1}{2}A)/\sqrt{3} = -1/\sqrt{3}$. Plugging A, B into the general solution gives us the particular solution $y = e^{x/2}(2\cos(\sqrt{3}x) - \dfrac{1}{\sqrt{3}}\sin(\sqrt{3}x))$.

Answer Key

1. Let a, b, c, d be real numbers. Compute the product $(a + bi)(c + di)$ and rewrite it in the form $x + yi$.

   $(ac - bd) + (ad + bc)i$

2. Rewrite the number $\sqrt{3} + i$ in the polar form $re^{i\theta}$.

   $2e^{i\pi/6}$

3. Rewrite the number $(-5 + 5i)^3$ in polar form.

   $250\sqrt{2}e^{i\pi/2}$

4. Write the number $8e^{5\pi/6}$ in the form $a + bi$.

   $-4\sqrt{3} + 4i$

5. Solve for the general solution to the differential equation $y'' - 6y' + 5 = 0$.

   $y = C_1e^{5x} + C_2e^{x}$

6. Solve for the general solution to the differential equation $y'' - 6y' + 9y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = -2$, $y'(0) = 1$.

   General solution: $y = C_1e^{3x} + C_2xe^{3x}$.

   Particular solution: $y = -2e^{3x} + 7xe^{3x}$.

7. Solve for the general solution to the differential equation $y'' - y' + y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = 2$, $y'(0) = 0$.

   General solution: $y = e^{x/2}(A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x))$.

   Particular solution: $y = e^{x/2}(2\cos(\sqrt{3}x) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}x))$.