Homework

1. Suppose there is a convergent power series representation for a solution y to the following differential equation in a neighborhood of 0. Using power series, compute the degree 5th polynomial approximation to the solution to the differential equation $y''+(1+x)y'+2x^2y=0$ with initial conditions $y(0)=1$, and $y'(0)=0$.

2. Suppose there is a convergent power series representation for a solution y to the following differential equation in a neighborhood of 0. Using power series, compute the degree 3 polynomial approximation (centered at 0) to the solution to the differential equation $(x^2-2)y''-3xy'+6y=0$ with initial conditions $y(0)=1$, and $y'(0)=1$.

3. Verify that the power series $\Sigma_0^\infty \frac{(-1)^n x^{2n}}{(2n)!}$ is a solution to the differential equation $y''+y=0$.

4. Compute the degree 3 polynomial approximation of the series solution to the differential equation $xy''+4y=0$ **centered at** $x=2$ with initial conditions $y(2)=4$, $y'(2)=2$. (i.e. $y=\Sigma_0^\infty b_n(x-2)^n$)

Solutions

1. Suppose there is a convergent power series representation for a solution y to the following differential equation in a neighborhood of 0. Using power series, compute the degree 5th polynomial approximation to the solution to the differential equation $y'' + (1+x)y' + 2x^2y = 0$ with initial conditions $y(0) = 1$, and $y'(0) = 0$.

   Solution: Let $y(x) = \Sigma_{n=0}^{\infty} b_n x^n$ be the power series solution to the IVP which is convergent near $x = 0$. Since $y(0) = 1$, $b_0 = 1$. From y, we can take the derivative to get y'. $y' = \Sigma_{n=1}^{\infty} n b_n x^{n-1}$. Since $y'(0) = 0$ and all the non-constant terms vanish at 0, $b_1 = 0$. To keep solving, we also need to know the formulas for y'', xy', and $2x^2y$. $y'' = \Sigma_{n=2}^{\infty} n(n-1)b_n x^{n-2}$. $x^2y = \Sigma_{n=0}^{\infty} b_n x^{n+2}$. $xy' = \Sigma_{n=1}^{\infty} n b_n x^n$.

   Since $y'' + y' + xy' + 2x^2y = 0$, the series obtained by summing the 4 power series must have 0 for each coefficient. The constant term of this sum is $2b_2 + b_1 + 0 + 0$ which come from y' and y''. Hence, $b_1 + 2b_2 = 0$. Since $b_1 = 0$, this means that $b_2 = 0$. Summing the coefficients of the linear terms of the 4 series gives us $6b_3 + 2b_2 + b_1 + 0 = 0$. Since $b_1 = b_2 = 0$, $b_3 = 0$. Summing the quadratic terms gives us $12b_4 + 3b_3 + 2b_2 + 2b_0 = 0$. Since $b_2 = b_3 = 0$, we have $b_4 = -2b_0/12 = -1/6$. Summing the cubic terms gives us $20b_5 + 4b_4 + 3b_3 + 2b_1 = 0$. Since $b_1 = b_3 = 0$, $b_5 = -4b_4/20 = -(-1/6)/5 = 1/30$. Now that we've found through b_5, we have the degree 5 polynomial approximation to the solution, which amounts to $y(x) \simeq 1 - (1/6)x^4 + (1/30)x^5$.

2. Suppose there is a convergent power series representation for a solution y to the following differential equation in a neighborhood of 0. Using power series, compute the degree 3 polynomial approximation (centered at 0) to the solution to the differential equation $(x^2 - 2)y'' - 3xy' + 6y = 0$ with initial conditions $y(0) = 1$, and $y'(0) = 1$.

   Solution: Let $y(x) = \Sigma_{n=0}^{\infty} b_n x^n$ be the power series solution to the IVP which is convergent near $x = 0$. Since $y(0) = 1$, $b_0 = 1$. From y, we can take the derivative to get y'. $y' = \Sigma_{n=1}^{\infty} n b_n x^{n-1}$. Since $y'(0) = 0$ and all the non-constant terms vanish at 0, $b_1 = 1$. To keep solving, we also need to know the formulas for $x^2y'', -2y'', -3xy'$, and $6y$.

$$
\begin{aligned}
x^2y'' &= \Sigma_{n=2}^{\infty} n(n-1)b_n x^n \\
-2y'' &= \Sigma_{n=2}^{\infty} - 2n(n-1)b_n x^{n-2} \\
-3xy' &= \Sigma_{n=1}^{\infty} - 3n b_n x^n \\
6y &= \Sigma_{n=0}^{\infty} 6 b_n x^n
\end{aligned}
$$

   Each coefficient to the resulting power series comprised by summing the above four power series must equal 0. Summing the constant terms gives us $0 + (-2(2)(1)b_2) + 0 + 6b_0 = -4b_2 + 6 = 0$. Hence, $b_2 = 6/4 = 3/2$. Summing the linear terms gives us $0 + (-2(3)(2)b_3) + (-3(1)b_1) + 6b_1 = -12b_3 + 3b_1 = -12b_3 + 3 = 0$. Hence, $b_3 = 1/4$.

Therefore, the degree 3 polynomial approximation to the power series is $y \simeq b_0 + b_1 x + b_2 x^2 + b_3 x^3 = 1 + x + (3/2)x^2 + (1/4)x^3$.

3. Verify that the power series $\Sigma_0^\infty \dfrac{(-1)^n x^{2n}}{(2n)!}$ is a solution to the differential equation $y'' + y = 0$.

   Set $y(x) = \Sigma_0^\infty \dfrac{(-1)^n x^{2n}}{(2n)!}$. Since this is a convergent series, we may compute it's derivative term-by-term and it will also be a convergent power series. Hence, $y' = \Sigma_1^\infty \dfrac{(-1)^n 2n x^{2n-1}}{(2n)!} = \Sigma_1^\infty \dfrac{(-1)^n x^{2n-1}}{(2n-1)!}$. To get y'' we take the derivative again, getting

$$\begin{aligned} y'' &= \Sigma_1^\infty \frac{(-1)^n (2n-1) x^{2n-2}}{(2n-1)!} \\ &= \Sigma_1^\infty \frac{(-1)^n x^{2n-2}}{(2n-2)!} \\ &= \Sigma_0^\infty \frac{(-1)^{n+1} x^{2n}}{(2n)!} \\ &= -\Sigma_1^\infty \frac{(-1)^n x^{2n}}{(2n)!} \\ &= -y \end{aligned}$$

   Hence, $y'' + y = (-y) + y = 0$.

4. Compute the degree 3 polynomial approximation of the series solution to the differential equation $xy'' + 4y = 0$ **centered at** $x = 2$ with initial conditions $y(2) = 4$, $y'(2) = 2$. (i.e. $y = \Sigma_0^\infty b_n (x-2)^n$)

   Before, we would just need xy'' and $4y$ and we would sum those series together. Since this series is centered at $x = 2$, We will want to find $(x-2)y''$, $2y''$, and $4y$ and will sum these together, knowing that their sum must equal the 0 function.

   Before we sum the individual series, we know from the initial conditions that $b_0 = 4$, $b_1 = 2$. We now only need to compute b_2 and b_3.

$$\begin{aligned} y &= \Sigma_0^\infty b_n (x-2)^n \\ y' &= \Sigma_1^\infty n b_n (x-2)^{n-1} \\ y'' &= \Sigma_2^\infty n(n-1) b_n (x-2)^{n-2} \\ 4y &= \Sigma_0^\infty 4 b_n (x-2)^n \\ (x-2)y'' &= \Sigma_2^\infty n(n-1) b_n (x-2)^{n-1} \\ 2y'' &= \Sigma_2^\infty 2n(n-1) b_n (x-2)^{n-2} \end{aligned}$$

   Summing the constant terms gives us $4b_0 + 0 + 2(2)(1)b_2 = 0$. Hence, $b_2 = -4b_0/4 = -4$. Summing the linear terms gives us $4b_1 + (2)(1)b_2 +$

$2(3)(2)b_3 = 0$. Hence, $b_3 = (-4(4) - 2(-4))/12 = -2/3$. Thus, together the first four terms of the series form $y \simeq 4 + 2(x-2) - 4(x-2)^2 - (2/3)(x-2)^3$.

Answer Key

1. Suppose there is a convergent power series representation for a solution y to the following differential equation in a neighborhood of 0. Using power series, compute the degree 5th polynomial approximation to the solution to the differential equation $y'' + (1+x)y' + 2x^2y = 0$ with initial conditions $y(0) = 1$, and $y'(0) = 0$.

   $y(x) \simeq 1 - (1/6)x^4 + (1/30)x^5$.

2. Suppose there is a convergent power series representation for a solution y to the following differential equation in a neighborhood of 0. Using power series, compute the degree 3 polynomial approximation (centered at 0) to the solution to the differential equation $(x^2 - 2)y'' - 3xy' + 6y = 0$ with initial conditions $y(0) = 1$, and $y'(0) = 1$.

   $y(x) \simeq = 1 + x + (3/2)x^2 + (1/4)x^3$.

3. Verify that the power series $\Sigma_0^\infty \dfrac{(-1)^n x^{2n}}{(2n)!}$ is a solution to the differential equation $y'' + y = 0$.

   Set $y(x) = \Sigma_0^\infty \dfrac{(-1)^n x^{2n}}{(2n)!}$. Since this is a convergent series, we may compute it's derivative term-by-term and it will also be a convergent power series. Hence, $y' = \Sigma_1^\infty \dfrac{(-1)^n 2n x^{2n-1}}{(2n)!} = \Sigma_1^\infty \dfrac{(-1)^n x^{2n-1}}{(2n-1)!}$. To get y'' we take the derivative again, getting

$$
\begin{aligned}
y'' &= \Sigma_1^\infty \frac{(-1)^n (2n-1) x^{2n-2}}{(2n-1)!} \\
&= \Sigma_1^\infty \frac{(-1)^n x^{2n-2}}{(2n-2)!} \\
&= \Sigma_0^\infty \frac{(-1)^{n+1} x^{2n}}{(2n)!} \\
&= -\Sigma_1^\infty \frac{(-1)^n x^{2n}}{(2n)!} \\
&= -y
\end{aligned}
$$

   Hence, $y'' + y = (-y) + y = 0$.

4. Compute the degree 3 polynomial approximation of the series solution to the differential equation $xy'' + 4y = 0$ **centered at** $x = 2$ with initial conditions $y(2) = 4$, $y'(2) = 2$. (i.e. $y = \Sigma_0^\infty b_n(x-2)^n$)

   $y \simeq 4 + 2(x_2) - 4(x-2)^2 - (2/3)(x-2)^3$.