Homework

1. A freshly made pie is at 400 degrees when it comes out of the oven. Assuming the temperature of the room is 70 degrees and it takes 10 minutes to cool to 200 degrees, how long must you wait to eat it if the hottest temperature your mouth can tolerate is 130 degrees? (Use Newton's law of cooling)

2. In order to clean your water while on a camping trip you must boil it for 5 minutes. If it takes 6 minutes to cool to 70 degrees, how much longer must you wait until the water is at 50 degrees Celsius if the air temperature at your campsite is 20 degrees Celsius? (Recall that the boiling temperature for water is 100 degrees Celsius.)

3. A tank contains 200 liters of a sugar and water solution with 10kg of dissolved sugar. If sugar water with a concentration of .04 kg per liter is being added to the tank at a rate of 10 liters per minute and the solution is being drained at the same rate, how much sugar in kg is in the water at time t?

4. A radioactive element, X, has a half-life of 25 years. Using the differential equation $dX/dt = kX$, and measuring t in years, compute what k must be.

5. A population of bacteria is modeled by the logistic equation $P' = kP(1 - P)$ where P is given as a percentage of some carrying capacity. Assuming that the initial population of bacteria is 10 percent the carrying capacity and at time $t = 10$ the population has risen to 20 percent, what percent of the carrying capacity will the population of bacteria be at time $t = 50$?

Solutions

1. A freshly made pie is at 400 degrees Fahrenheit when it comes out of the oven. In a 70 degree room, how long must you wait to eat it if the hottest temperature your mouth can tolerate is 130 degrees? (Use Newton's law of cooling)

   Solution: Let t represent time and let $T(t)$ represent the temperature of the pie as a function of t. Newton's law of cooling states that $T'(t) = k(T - T_{env})$ where T_{env} is the ambient temperature and k is a constant. This is a separable equation. Separating variables, we get $\frac{dT}{T - T_{env}} = kdt$. Integrating this, we have

$$\int \frac{dT}{T - T_{env}} = \int kdt$$
$$\ln |T - T_{env}| = kt + C_1$$
$$e^{\ln |T - T_{env}|} = e^{kt + C_1}$$
$$|T - T_{env}| = Ce^{kt}$$
$$T = Ce^{kt} + T_{env}$$

   Plugging in our variables, we get $T = Ce^{kt} + 70$. Since $T(0) = 400$, we have $400 = Ce^0 + 70 = C + 70$. Hence, $C = 330$. To get k, we use the information that $T(10) = 200$. So, $200 = 330e^{10k} + 70$. Simplifying gets us $130/330 = e^{10k}$. So $k = \ln(130/330)/10 \simeq -.093$. Now, we want to know at what time, t, $T(t) = 130$. We solve the equation $130 = 330e^{-.093t} + 70$ for t. In steps, we get $(130 - 70)/330 = e^{-.093t}$. Taking logs, we get $\ln(60/330) = -.093t$ and using a calculator we see that $t \simeq 25.7$ minutes.

2. In order to clean your water while on a camping trip you must boil it for 5 minutes. If it takes 6 minutes to cool to 70 degrees, how much longer must you wait until the water is at 50 degrees Celsius if the air temperature at your campsite is 20 degrees Celsius? (Recall that the boiling temperature for water is 100 degrees Celsius.)

   SOlution: We first assign variables. Let T represent temperature of the water as a function of time t. $T_{env} = 20$, the temperature of the air. Newton's law of cooling states that $\frac{dT}{dt} = k(T - T_{env})$. Separating variables, integrating, and simplifying as in the previous problem, we get $T = Ce^{kt} + T_{env}$. Using the initial condition that at time 0, $T = 100$, we get $100 = Ce^0 + 20 = C + 20$. Hence, $C = 80$. We then solve for k using the other time point which says that $T(6) = 70$. Plugging in, we get $70 = 80e^{6k} + 20$. Hence, $k = \ln(\frac{70 - 20}{80})/6 \simeq -.078$. We now solve for the time t at which $T = 50$. We have $50 = 80e^{-.078t} + 20$. Solving for t, we get $t \simeq \ln((50 - 20)/80)/(-.078) \simeq 12.5$ minutes. Hence, it takes an *additional* 6.5 minutes to cool to 50 degrees from the time when it was 70 degrees.

3. A tank contains 200 liters of a sugar and water solution with 10kg of dissolved sugar. If sugar water with a concentration of .04 kg per liter is being added to the tank at a rate of 10 liters per minute and the solution is being drained at the same rate, how much sugar in kg is in the water at time t?

   Solution: A mixing problem such as this follows the formula $m' = c_{in}r_{in} - c_{out}r_{out}$ where m will represent the total mass of the sugar in the tank, c_{in} is the concentration of sugar in the water coming in, r_{in} is the rate at which the solution is coming in, and c_{out}, r_{out} are defined similarly for the outgoing solution. Using the information given in the problem, we have that $m(0) = 200$, $c_{in} = .04$kg/L, $r_{in} = 10$L/min, $c_{out} = m/200$kg/L which is the concentration of the water in the tank, and $r_{out} = 10$L/min. Putting this together, we have $\frac{dm}{dt} = .4 - m/20 = (8-m)/20$. Separating variables, we get $\dfrac{dm}{m-8} = -1/20dt$. Integrating gives us $\ln|m-8| = -t/20 + C_1$. Exponentiating and then adding 8, we get $m = 8 + e^{-t/20+C_1} = 8 + Ce^{-t/20}$. We can solve for C by plugging in our initial condition, which states that at time 0, the mass is 10kg. Hence, $10 = 8 + Ce^0$ and so $C = 2$. Hence, the total mass of sugar in the tank is given by $m = 8 + 2e^{-t/20}$.

4. A radioactive element, X, has a half-life of 25 years. Using the differential equation $dX/dt = kX$, and measuring t in years, compute what k must be.

   Solution: We first solve the DE using separation of variables. $\int dX/X = \int k dt$. Integrating, we get $\ln|X| = kt + C_1$. Exponentiating, we get $X = Ce^{kt}$. At time $t = 0$, $X(0) = C$. At time $t = 25$, $X(25) = X(0)/2 = C/2$. Hence, $C/2 = Ce^{25k}$. Now we can divide both sides by C and we get $1/2 = e^{25k}$. Taking logs, we get $\ln(1/2) = 25k$. And so $k \simeq -.0277$.

5. A population of bacteria is modeled by the logistic equation $P' = kP(1-P)$ where P is given as a percentage of some carrying capacity. Assuming that the initial population of bacteria is 10 percent the carrying capacity and at time $t = 10$ the population has risen to 20 percent, what percent of the carrying capacity will the population of bacteria be at time $t = 50$?

   Separating variables, we get $\dfrac{dP}{P(1-P)} = kdt$. Before we can integrate, we must first do a partial fraction decomposition of the left side of the equation. Doing so gets us $(\frac{1}{P} + \frac{1}{1-P})dP = kdt$. Integrating, we get $\ln P - \ln(1-P) = kt + C_1$. We omit absolute value signs because P is assumed to be between 0 and 1. We can combine the left side of the equation to get $\ln(\dfrac{P}{1-P})$. Then, exponentiating, we get $\dfrac{P}{1-P} = Ce^{kt}$. Plugging in the point $(0, 0.1)$, we get $\dfrac{0.1}{0.9} = Ce^0 = C$. Hence, $C = 1/9$.

Plugging in the point $(10, 0.2)$, we get $\dfrac{0.2}{0.8} = e^{10k}/9$. Hence $\ln(9/4) = 10k$. And so $k \simeq .081$. Multiplying by $1-P$ to both sides, distributing and then moving all values with P to the left side gets us $P(1+e^{kt}/9) = e^{kt}/9$. We then have that $P \simeq \dfrac{e^{.081t}/9}{1+e^{.081t}/9}$. At time $t = 50$, $P \simeq \dfrac{e^{.081(50)}/9}{1+e^{.081(50)}/9} \simeq 0.86$. That is at time $t = 50$, the population will have reached 86 percent of its carrying capacity.

Answer Key

1. A freshly made pie is at 400 degrees when it comes out of the oven. Assuming the temperature of the room is 70 degrees and it takes 10 minutes to cool to 200 degrees, how long must you wait to eat it if the hottest temperature your mouth can tolerate is 130 degrees? (Use Newton's law of cooling)

   25.7 minutes

2. In order to clean your water while on a camping trip you must boil it for 5 minutes. If it takes 6 minutes to cool to 70 degrees, how much longer must you wait until the water is at 50 degrees Celsius if the air temperature at your campsite is 20 degrees Celsius? (Recall that the boiling temperature for water is 100 degrees Celsius.)

   6.5 minutes

3. A tank contains 200 liters of a sugar and water solution with 10kg of dissolved sugar. If sugar water with a concentration of .04 kg per liter is being added to the tank at a rate of 10 liters per minute and the solution is being drained at the same rate, how much sugar in kg is in the water at time t?

   $m = 8 + 2e^{-t/20}$

4. A radioactive element, X, has a half-life of 25 years. Using the differential equation $dX/dt = kX$, and measuring t in years, compute what k must be.

   $k \simeq -.0277$

5. A population of bacteria is modeled by the logistic equation $P' = kP(1 - P)$ where P is given as a percentage of some carrying capacity. Assuming that the initial population of bacteria is 10 percent the carrying capacity and at time $t = 10$ the population has risen to 20 percent, what percent of the carrying capacity will the population of bacteria be at time $t = 50$?

   86 percent