Homework

1. Verify (by plugging in) that $y = e^{-3x}+2x+3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

2. Verify that $y = \pi e^{-\cos x}$ is a solution to the differential equation $y' = y \sin x$.

3. Find the general solution to the differential equation $y' = \sin x + x$.

4. Solve for the particular solution to the differential equation $y' = 2x \cos(x^2)$ satisfying $y(0) = 3$. (Recall that a problem of this form is called an initial value problem.)

5. Draw a slope field for all integer coordinates (x, y) with $-3 \leq x, y \leq 3$ for the following differential equations:

   (a) $y' = y - x$

   (b) $y' = 3y + xy$

6. Use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$ where $y(x)$ is a solution to the differential equation $y' + 3y = x^2$ passing through the point $(0, 2)$.

7. For the initial value problem with differential equation $yy' = x$ and initial condition $y(1) = 1$, estimate $y(1.4)$ using Euler's method with step size $h = 0.1$.

Solutions

1. Verify (by plugging in) that $y = e^{-3x}+2x+3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

   Solution: In order to verify that $y = e^{-3x} + 2x + 3$ is a solution, we must first take the derivative to get y'. Doing so, we get $y' = -3e^{-3x} + 2$. Plugging in our values for y and y', we get

$$\begin{aligned} y' + 3y &= (-3e^{-3x} + 2) + 3(e^{-3x} + 2x + 3) \\ &= -3e^{-3x} + 3e^{-3x} + 6x + 2 + 9 \\ &= 6x + 11. \end{aligned}$$

   And we are done.

2. Verify that $y = \pi e^{-\cos x}$ is a solution to the differential equation $y' = y \sin x$.

   Solution: In order to verify that $y = \pi e^{-\cos x}$ is a solution, we must first take it's derivative to get y'. Doing so, we get $y' = \pi e^{-\cos x}(\sin x)$. Plugging in our values for y, y', we get

$$\begin{aligned} y' &= \pi e^{-\cos x} \sin x \\ &= y \sin x. \end{aligned}$$

   And we are done.

3. Find the general solution to the differential equation $y' = \sin x + x$.

   Solution: Since x is the only variable in the differential equation, we may integrate both sides with respect to x as follows:

$$\begin{aligned} y' &= \sin x + x \\ \int y' dx &= \int \sin x + x dx \\ y &= -\cos x + x^2/2 + C. \end{aligned}$$

   And we are done.

4. Solve for the particular solution to the differential equation $y' = 2x\cos(x^2)$ satisfying $y(0) = 3$. (Recall that a problem of this form is called an initial value problem.)

   Solution: To find the particular solution, we start by solving for the general solution. The method here is the same as for the previous problem.

$$\begin{aligned} y' &= 2x\cos(x^2) \\ \int y'dx &= \int 2x\cos(x^2)dx \\ &= \int \cos(u)du \qquad\qquad (u = x^2) \\ y &= \sin u + C \\ &= \sin(x^2) + C \end{aligned}$$

We now solve for the correct value of C which satisfies the initial condition. Plugging in the initial condition, we have

$$3 = y(0) = \sin(0^2) + C = 0 + C = C.$$

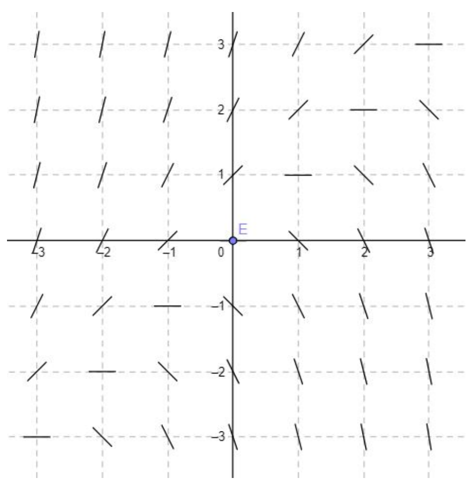
Hence, $C = 3$ giving us the particular solution $y = \sin(x^2) + 3$.

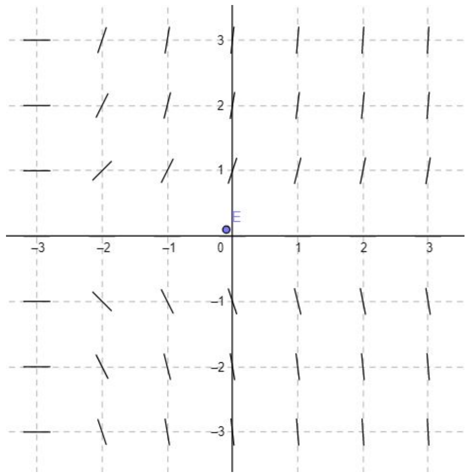
5. Draw a slope field for all integer coordinates (x, y) with $-3 \le x, y \le 3$ for the following differential equations:

   (a) $y' = y - x$

   (b) $y' = 3y + xy$

   Solution: At each point with x, y integers between -3 and 3, compute the slope according to the function $y' = f(x, y)$ and plot it on a graph as seen in the diagrams below.

   (a) 



(b)

6. Use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$ where $y(x)$ is a solution to the differential equation $y' + 3y = x^2$ passing through the point $(0, 2)$.

   Solution: To estimate $y(0.3)$ with a step size of $h = 0.1$ starting at 0 requires taking 3 steps. We first express y' as a function $f(x, y)$. Since $y' + 3y = x^2$, we have $y' = x^2 - 3y = f(x, y)$. To begin the Euler's method algorithm, we begin with $x_0 = 0, y_0 = 2$.

$$x_1 = X_0 + h = 0 + 0.1 = 0.1$$
$$y_1 = y_0 + h * f(x_0, y_0) = 2 + 0.1 * (0 - 3 * 2) = 1.4$$
$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$
$$y_2 = y_1 + h * f(x_1, y_1) = 1.4 + 0.1 * (0.1^2 - 3 * 1.4) = .981$$
$$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$$
$$y_3 = y_2 + h * f(x_2, y_2) = .981 + 0.1 * (0.2^2 - 3 * .981) = .6907$$

   We have thus found $y(0.3) \simeq y_3 = .6907$ and we are done.

7. For the initial value problem with differential equation $yy' = x$ and initial condition $y(1) = 1$, estimate $y(1.4)$ using Euler's method with step size $h = 0.1$.

   To estimate $y(1.4)$ with a step size of $h = 0.1$ starting at 1 requires taking 4 steps. We first express y' as a function $f(x, y)$. Since $yy' = x$, we have $y' = x/y = f(x, y)$. To begin the Euler's method algorithm, we begin with

$x_0 = 1, y_0 = -3.$

$$
\begin{aligned}
&x_1 = X_0 + h = 1 + 0.1 = 1.1\\
&y_1 = y_0 + h * f(x_0, y_0) = -3 + 0.1 * (1/-3) \simeq -3.03333\\
&x_2 = x_1 + h = 1.1 + 0.1 = 1.2\\
&y_2 = y_1 + h * f(x_1, y_1) \simeq -3.03333 + 0.1 * (1.1/-3.03333) \simeq -3.0696\\
&x_3 = x_2 + h = 1.2 + 0.1 = 1.3\\
&y_3 = y_2 + h * f(x_2, y_2) \simeq -3.0696 + 0.1 * (1.2/-3.0696) \simeq -3.1087\\
&x_4 = x_3 + h = 1.3 + 0.1 = 1.4\\
&y_4 = y_3 + h * f(x_2, y_2) \simeq -3.1087 + 0.1 * (1.3/-3.1087) \simeq -3.1505
\end{aligned}
$$

We have thus found $y(1.4) \simeq y_4 \simeq -3.1505$ and we are done.

Answer Key

1. Verify (by plugging in) that $y = e^{-3x}+2x+3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

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2. Verify that $y = \pi e^{-\cos x}$ is a solution to the differential equation $y' = y \sin x$.

$$\begin{aligned} y' &= \pi e^{-\cos x} \sin x \\ &= y \sin x. \end{aligned}$$

3. Find the general solution to the differential equation $y' = \sin x + x$.

   $y = -\cos x + x^2/2 + C.$

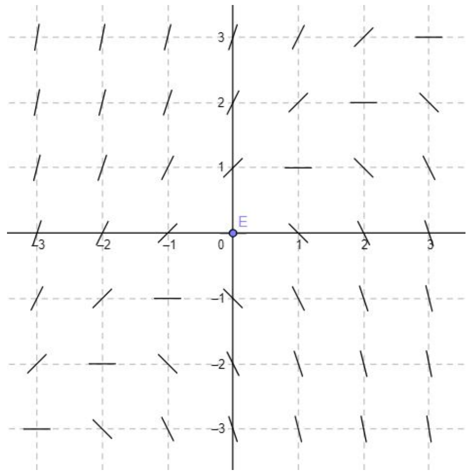
4. Solve for the particular solution to the differential equation $y' = 2x\cos(x^2)$ satisfying $y(0) = 3$. (Recall that a problem of this form is called an initial value problem.)

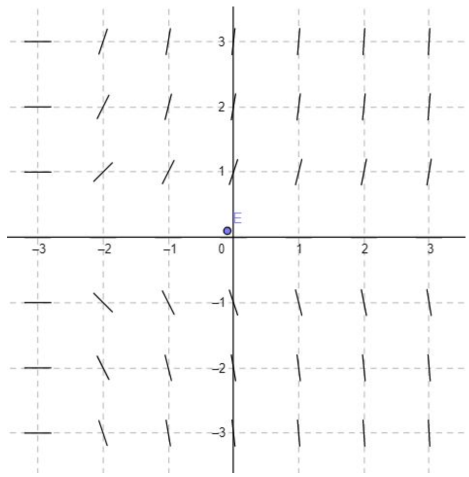
   $y = \sin(x^2) + 3.$

5. Draw a slope field for all integer coordinates (x, y) with $-3 \leq x, y \leq 3$ for the following differential equations:

   (a) $y' = y - x$

   (b) $y' = 3y + xy$

(a)


(b)


6. Use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$ where $y(x)$ is a solution to the differential equation $y' + 3y = x^2$ passing through the point $(0, 2)$.

   $y(0.3) \simeq .6907$

7. For the initial value problem with differential equation $yy' = x$ and initial condition $y(1) = 1$, estimate $y(1.4)$ using Euler's method with step size $h = 0.1$.

   $y(1.4) \simeq -3.1505$