MAT 127 HW 20-25

1. Problems

1. Find the Maclaurin series of $F(x)$ by integrating the Maclaurin series of the the integrand.
$$F(x) = \int_0^x e^{t^4}\,dt.$$
2. Determine if the following series converge.
$$\sum_{n=3}^{\infty} \frac{1}{2n\ln(n)}.$$
3. Determine if the following series converge.
$$\sum_{n=3}^{\infty} 2^{-\ln(n)}.$$
4. Determine if the following series converge.
$$\sum_{n=3}^{\infty} \frac{\cos(n)+2}{n^3}.$$
5. Determine if the following series converge.
$$\sum_{n=1}^{\infty} \frac{n!}{(n+3)!}.$$

2. Answer Key

1. $\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)n!}$.
2. Diverges.
3. Converges.
4. Converges.
5. Converges.

3. Solutions

1. Consider $F(x) = \int_0^x e^{t^4}dt$. And note that the Maclaurin series of $f(x) = e^{-t^4}$ is $\sum_{n=0}^{\infty} \frac{t^{4n}}{n!}$. Thus, the Maclaurin series of F is
$$\int_0^x \sum_{n=0}^{\infty} \frac{t^{4n}}{n!}dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^{4n}}{n!}dt = \sum_{n=0}^{\infty} \frac{t^{4n+1}}{(4n+1)n!}\Big|_0^x = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)n!}.$$
2. We will use the integral test since $f(x) = \frac{1}{2x\ln(x)}$ on $[3,\infty)$ satisfies the hypotheses of this test. So consider (We will the u-substitution $u = \ln(x)$ so $du = \frac{dx}{x}$).
$$\int_3^{\infty} \frac{1}{2x\ln(x)}dx = \frac{1}{2}\int_3^{\infty} \frac{1}{\ln(x)}\frac{dx}{x} = \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{1}{u}du = \frac{1}{2}\lim_{N\to\infty} \ln(N) - \ln(\ln(3)).$$

Thus we see that this integral diverges so the sum diverges.

3. Consider:
$$\sum_{n=3}^{\infty} 2^{-\ln(n)} = \sum_{n=3}^{\infty} \frac{1}{2^{\ln(n)}} = \sum_{n=3}^{\infty} \frac{1}{e^{\ln(2)\ln(n)}} = \sum_{n=3}^{\infty} \frac{1}{n^{\ln(2)}}.$$
And as $\ln(2) > 1$ we have by the p-series test that this sum converges.
4. Consider:
$$\sum_{n=3}^{\infty} \frac{\cos(n)+2}{n^3}.$$
Note that $\cos(n)+2 \leq 3$ so $\frac{\cos(n)+2}{n^3} \leq \frac{3}{n^3}$. And by the p-test we know that $\sum_{n=3}^{\infty} \frac{1}{n^3}$ converges so by comparison we have $\sum_{n=3}^{\infty} \frac{\cos(n)+2}{n^3}$ converges.
5. Consider"
$$\sum_{n=1}^{\infty} \frac{n!}{(n+3)!} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}.$$
And note that
$$(n+1)(n+2)(n+3) = n^3 + 6n^2 + 11n + 6.$$
So for all $n \leq 1$
$$n^3 + 6n^2 + 11n + 6 > n^3.$$
And so
$$\frac{1}{(n+1)(n+2)(n+3)} \leq \frac{1}{n^3}.$$
And by the p-test we know that $\sum_{n=3}^{\infty} \frac{1}{n^3}$ converges so by comparison we have $\sum_{n=1}^{\infty} \frac{n!}{(n+3)!}$ converges.