MAT 126 HW 24-27

1. Problems

1. Suppose that when a particle lies on a line x meters from the origin, the force exerted on it is $x + \frac{1}{x}$ Newtons. How much work done to move the particle from the point $x = 2$ to the point $x = 4$?
2. Suppose a spring has a natural length of 10cm and it takes a force of 30 Newtons to stretch it to a length of 20cm. How much work is needed to stretch the spring to 60cm?
3. Suppose it takes a force of 25 Newtons to stretch a spring from 10m to 35m. It's a big spring. How much work needs to be done in order to make it stretch from 10m to 15m?
4. Suppose we have a tank in the shape of a cone with the point facing downward. The radius of the top of the tank is 6 meters and the height is 5 meters. Suppose the tank is filled to a height of 3 meters with water. How much work will it take to pump all of the water out of the top of the tank? Recall that the density of water is $\rho = 1000\text{kg/m}^3$. Assume that the acceleration due to gravity is 10m/s^2.
5. Suppose a disk of radius 3m is vertically submerged in water so that the top of the disk is 10m below the water. Set up, but do not evaluate, the integral to calculate the total hydrostatic force on the disk. Recall that the density of water is $\rho = 1000\text{kg/m}^3$. Assume that the acceleration due to gravity is 10m/s^2.

2. Answer Key

1. $6 + \ln 2$ J
2. 26.25 J
3. 125/2 J
4. 38400π J
5. $\int_{10}^{16} 20000y\sqrt{9-(y-13)^2}\,dy$

3. Solutions

1. Work is the integral of force over distance, so the work in Joules is
$$W = \int_2^4 x + \frac{1}{x}\, dx = \frac{1}{2}x^2 + \ln|x| \Big|_2^4 = 8 + \ln 4 - 2 - \ln 2 = 6\ln 2.$$

2. Using the expression $f(x) = kx$ for the force required to stretch a spring a distance of x, the given information tells us that $30 = k(0.2)$, or $k = 150$. Hence the work is the integral of $150x$ over the interval 0.1 to 0.6.
$$W = \int_{0.1}^{0.6} 150x\, dx = 75x^2 \Big|_{0.1}^{0.6} = 26.25 \text{ J}.$$

3. The given information tells us that $25 = k(25)$, or the spring constant k is 1 in this case. Hence the work in Joules is
$$W = \int_{10}^{15} x\, dx = \frac{1}{2}x^2 \Big|_{10}^{15} = \frac{125}{2}.$$

4. We must first find the volume of a small horizontal cross section of the cone at depth x. These cross sections are small cylinders with height dx and they lie $5 - x$ meters above the bottom of the tank. The radius $r(x)$ can be found by noting that the triangle formed by the radius and the distance of the cross section to the bottom is similar to the triangle formed by the radius of the cone and the height of the cone. Hence $r(x)/(5 - x) = 6/5$, or $r(x) = 6(5 - x)/5$. Thus the volume of this cylindrical cross section is $V(x) = \pi r(x)^2\, dx = (36/25)\pi(5 - x)^2\, dx$. The mass is then given by $1000V(x) = 1440\pi(5-x)^2\, dx$ and the force required to move this small cylinder is $gm = 14400\pi(5 - x)^2\, dx$. Finally, we integrate this quantity over the necessary depths (x goes from 3 meters to 5 meters), we get the work, in Joules, as
$$W = 14400\pi \int_3^5 (5 - x)^2\, dx = 14400\pi \left(25x - 5x^2 + \frac{1}{3}x^3\right) \Big|_3^5 = 38400\pi$$

5. At each point, the pressure applies is equal to $P = \rho g y$ where ρ is the density of the water, g is the gravitational constant, and y is the depth of a point on the disk. This comes out to $P = 10000y$. Next we need to integrate this over the area of the disk. To do this we cut the disk horizontally and find the area of the cross section. At a depth of y, the length of the cross section is given by $2\sqrt{9 - (y - 13)^2}$ and the height is dy. So the total area of the cross section is $2\sqrt{9 - (y - 13)^2}\, dy$. Hence the total pressure on this small cross section is $P \cdot 2\sqrt{9 - (y - 13)^2}\, dy = 20000y\sqrt{9 - (y - 13)^2}\, dy$. Finally, we integrate this over the total depth of the disk, which gives the final integral representing force as
$$F = \int_{10}^{16} 20000y\sqrt{9 - (y - 13)^2}\, dy.$$