MAT 126 HW 16-20

1. Problems

1. Calculate the volume obtained by revolving the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = 9 - x^2$ about the x-axis.
2. Calculate the volume obtained by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$, $g(x) = e^{x-4}$, and the lines $x = 0$ and $x = 3$ about the x-axis.
3. Calculate the volume obtained from the same region as in Problem 2 but instead revolve about the line $y = -2$.
4. Find the volume of the solid bounded by the curve $y = \sqrt{9 - x^2}$ and whose cross sections are squares.
5. Find the volume of the solid bounded by $f(x) = \sin x$ and the lines $x = 0$ and $x = \pi$ and whose cross sections are equilateral triangles.

2. Answer Key

1.
$$2\pi\left(\frac{352}{3}\right)$$

2.
$$\frac{\pi}{2}\left(9-e^{-2}+e^{-8}\right)$$

3.
$$\frac{\pi}{2}\left(9+16\sqrt{3}-e^{-2}-8e^{-1}+e^{-8}+8e^{-4}\right)$$

4. 36

5.
$$\frac{\pi\sqrt{3}}{8}$$

3. Solutions

1. Graphing, we see that $g(x)$ lies above $f(x)$. To solve for the endpoints of our interval of integration, we write $9-x^2 = x^2+1$. Solving for x yields $x = \pm 2$. Using the washer method, the volume is given by
$$\begin{aligned}\pi \int_{-2}^{2} (9-x^2)^2 - (x^2+1)^2 \, dx &= \pi \int_{-2}^{2} 80 - 16x^2 \, dx \\ &= \pi \left(80x - \frac{16}{3}x^3 \right) \Big|_{-2}^{2} = 2\pi \left(\frac{352}{3} \right).\end{aligned}$$
2. Graphing, we see that $f(x)$ lies above $g(x)$. Using the washer method, the volume is given by
$$\begin{aligned}\pi \int_{0}^{3} \left(\sqrt{x}\right) - \left(e^{x-4}\right) \, dx &= \pi \int_{0}^{3} x - e^{2x-8} \, dx \\ &= \frac{\pi}{2} \left(x^2 - e^{2x-8}\right) \Big|_{0}^{3} = \frac{\pi}{2} \left(9 - e^{-2} + e^{-8}\right)\end{aligned}$$
3. Much of the setup is as in the previous solution, however now our radii are given by $f(x)+2$ and $g(x)+2$. Using these in the washer method, the volume is given by
$$\begin{aligned}\pi \int_{0}^{3} \left(\sqrt{x}+2\right) - \left(e^{x-4}+2\right) \, dx &= \pi \int_{0}^{3} x + 4\sqrt{x} - e^{2x-8} - 4e^{x-4} \, dx \\ &= \frac{\pi}{2} \left(x^2 + \frac{16}{3} - \frac{1}{2}e^{2x-8} - 4e^{x-4} \right) \Big|_{0}^{3} \\ &= \frac{\pi}{2} \left(9 + 16\sqrt{3} - e^{-2} - 8e^{-1} + e^{-8} + 8e^{-4} \right)\end{aligned}$$
4. The area of each cross section is given by $(\sqrt{9-x^2})^2 = 9-x^2$. The volume is given by the integral of this expression over the interval $[-3, 3]$. We get
$$\int_{-3}^{3} 9 - x^2 \, dx = 9x - \frac{1}{3}x^3 \Big|_{-3}^{3} = 36.$$
5. The area of each cross section is given by $\frac{\sqrt{3}}{4}\sin^2 x$. The volume is given by the integral of this expression over the interval $[0, \pi]$. We get
$$\frac{\sqrt{3}}{4} \int_{0}^{\pi} \sin^2 x \, dx = \frac{\sqrt{3}}{8} \int_{0}^{\pi} 1 - \cos 2x \, dx = \frac{\sqrt{3}}{8} \left(x - \frac{1}{2}\sin 2x \right) \Big|_{0}^{\pi} = \frac{\pi\sqrt{3}}{8}.$$