MAT 126 HW 15

1. Problems

1. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_{10}^{\infty} -\frac{4}{x^5}\, dx$$
2. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_{0}^{3} \frac{1}{2x-2}\, dx$$
3. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_{4}^{7} \frac{9}{\sqrt{x-4}}\, dx$$
4. Find the area of the bounded region contained between the curves $f(x) = x^3$ and $g(x) = \sqrt{x}$.
5. Find the area of the bounded region contained between the curves $f(x) = \sin(\frac{1}{2}x)$ and $g(x) = (\frac{1}{2}x - 1)^2$ and between the lines $x = \pi/2$ and $x = \pi$.

2. Answer Key

1. The integral converges to $-1/10^4$.
2. The integral diverges.
3. The integral converges to $2\sqrt{3}$.
4. The area is $5/12$
5. The area is
$$-\frac{2}{3}\left(\frac{\pi}{2}-1\right)^3+\sqrt{2}+\frac{2}{3}\left(\frac{\pi}{4}-1\right)^3$$

3. Solutions

1. The integral converges because it is a p-integral with $p = 5 > 1$. Explicitly calculating we see

$$\int_{10}^{\infty} \frac{-4}{x^5}\,dx = \lim_{c\to\infty} -4\int_{10}^{c} \frac{1}{x^5}\,dx$$
$$= \lim_{c\to\infty} -4\left(-\frac{1}{4x^4}\right)\Big|_{10}^{c} = \lim_{c\to\infty} \frac{1}{c^4} - \frac{1}{10^4} = -\frac{1}{10^4}.$$

2. The integral diverges. There's a discontinuity at $x = 1$ so we must write

$$\int_0^3 \frac{1}{2x-2}\,dx = \int_0^1 \frac{1}{2x-2}\,dx + \int_1^3 \frac{1}{2x-2}\,dx.$$

By definition, the integral converges if and only if both of the integrals on the right hand side converge, so we will only show that the first one diverges (since this will be enough to show that the whole integral diverges). We have

$$\int_0^1 \frac{1}{2x-2}\,dx = \lim_{c\to 1^-} \int_0^c \frac{1}{2x-2}\,dx = \lim_{c\to 1^-} \frac{1}{2}\ln|x-1|\Big|_0^c = \frac{1}{2}\left(\lim_{c\to 1^-} \ln|c-1|\right)$$

since $\ln|-1| = \ln(1) = 0$. But the limit on the right hand side is $-\infty$. Hence the limit does not exist, so the integral diverges.

3. The integral converges. We have

$$\int_4^7 \frac{9}{\sqrt{x-4}}\,dx = \lim_{c\to 4^+} \int_c^7 \frac{9}{\sqrt{x-4}}\,dx$$
$$= \lim_{c\to 4^+} 2\sqrt{x-4}\,\Big|_c^7 = 2\sqrt{3} \lim_{c\to 4^+} 2\sqrt{c-4} = 2\sqrt{3}$$

4. The curves meet at $x = 0$ and $x = 1$, with the graph of $\sqrt{x}$ lying above that of x^3. So the area we are searching for is

$$\int_0^1 \sqrt{x} - x^3\,dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^4\,\Big|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$

5. Graphing the functions on a plane, one sees that $f(x)$ lies above $g(x)$ on the specified interval $[\pi/2, \pi]$, so the area we are searching for is

$$\int_{\frac{\pi}{2}}^{\pi} \sin\left(\frac{1}{2}x\right) - \left(\frac{1}{2}x - 1\right)^2\,dx = -2\cos\left(\frac{1}{2}x\right) - \frac{2}{3}\left(\frac{1}{2}x-1\right)^3\,\Big|_{\frac{\pi}{2}}^{\pi}$$
$$= -\frac{2}{3}\left(\frac{\pi}{2} - 1\right)^3 + \sqrt{2} + \frac{2}{3}\left(\frac{\pi}{4} - 1\right)^3$$