MAT 126 HW 10-11

1. Problems

1. Compute the following indefinite integral:
$$\int \tan x \; dx$$
2. Compute the following indefinite integral:
$$\int \sin^4(x) \; dx$$
3. Compute the following indefinite integral:
$$\int \tan^4(x) \sec^2(x) \; dx$$
4. Compute the following indefinite integral:
$$\int \sin^4(x) \cos^5(x) \; dx$$
5. Compute the following indefinite integral using trig substitution:
$$\int \sqrt{1 - 9x^2} \; dx$$

2. Answer Key

1.
$$-\ln|\cos x| + C$$

2.
$$\frac{1}{4}\left(\frac{3x}{2} - \sin(2x) + \frac{1}{8}\sin(4x)\right) + C$$

3.
$$\frac{1}{5}\tan^5(x) + C$$

4.
$$\frac{1}{5}\sin^5(x) - \frac{2}{7}\sin^7(x) + \frac{1}{9}\sin^9(x) + C$$

5.
$$-\frac{1}{6}\arccos(3x) + \frac{1}{6}(3x)(\sqrt{1-9x^2})$$

3. Solutions

1. We write $\tan x = \sin x/\cos x$ and perform u-substitution with $u = \cos x$. This gives

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|\cos x| + C.$$

2. We use the identities $\cos^2(x) = (1+\cos(2x))/2$ and $\sin^2(x) = (1-\cos(2x))/2$. Note that the second identity gives us

$$\sin^4(x) = \left[\sin^2(x)\right]^2 = \left[(1-\cos(2x))/2\right]^2.$$

It follows that

$$\begin{aligned}
\int \sin^4(x) \, dx &= \int \left[\frac{1-\cos(2x)}{2}\right]^2 \, dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x) \, dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x) \, dx \\
&= \frac{1}{4}\left(x - \sin(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x)\right) + C \\
&= \frac{1}{4}\left(\frac{3x}{2} - \sin(2x) + \frac{1}{8}\sin(4x)\right) + C
\end{aligned}$$

3. This is a substitution with $u = \tan x$. Then $du = \sec^2 x \, dx$ and we have

$$\int \tan^4(x)\sec^2(x) \, dx = \int u^4 \, du = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5(x) + C.$$

4. Write $\cos^5(x) = \cos^4(x)\cos x = (1-\sin^2(x))^2\cos x$. Then we can perform a substitution with $u = \sin x$ which gives us

$$\begin{aligned}
\int \sin^4(x)\cos^5(x) \, dx &= \int u^4(1-u^2)^2 \, du \\
&= \int u^4 - 2u^6 + u^8 \, du \\
&= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{5}\sin^5(x) - \frac{2}{7}\sin^7(x) + \frac{1}{9}\sin^9(x) + C.
\end{aligned}$$

5. This requires a trigonometric substitution. We will use $3x = \cos\theta$ (although using $3x = \sin\theta$ would work just as well). This gives us

$$\int \sqrt{1-9x^2} \, dx = -\frac{1}{3}\int \sin\theta\sqrt{1-\cos^2\theta} \, d\theta = -\frac{1}{3}\int \sin^2\theta \, d\theta.$$

Using the identity $\sin^2(x) = (1-\cos(2x))/2$ one finds that $\int \sin^2\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$. Now we use the identity $\sin 2\theta = 2\sin\theta\cos\theta$ which gives us

$$\int \sqrt{1-9x^2} \, dx = -\frac{1}{3}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) = -\frac{1}{3}\left(\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta\right).$$

Finally, getting everything back in terms of cosine gives us

$$\begin{aligned}\int \sqrt{1-9x^2}\,dx &= -\frac{1}{3}\left(\frac{1}{2}\theta - \frac{1}{2}\left(\sqrt{1-\cos^2\theta}\right)\cos\theta\right)\\ &= -\frac{1}{3}\left(\frac{1}{2}\arccos(3x) - \frac{1}{2}\left(\sqrt{1-9x^2}\right)(3x)\right)\\ &= -\frac{1}{6}\arccos(3x) + \frac{1}{6}(3x)(\sqrt{1-9x^2}).\end{aligned}$$