MAT126 Homework 4-5

Problems

1. Compute the following limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$$

   Hint: The limit can be expressed as a definite integral.

2. Compute the following definite integral:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 6\csc(x)\cot(x)\ dx$$

3. Compute the following definite integral:

$$\int_1^2 \frac{6}{x} + \frac{\sqrt[3]{x^2}}{2} - \frac{1}{2x^3}\, dx$$

4. Compute the following definite integral:

$$\int_0^3 f(x)\ dx$$

   where

$$f(x) = \begin{cases} 2x & x > 1 \\ 4x^3 - 3x^2 & x \le 1 \end{cases}$$

5. If the velocity of an object is given by

$$v(t) = -9.8t + 10,$$

   find the total displacement of the object after 2 seconds.

6. If

$$f(x) = \int_1^{cos(x)} t\sin(t) + \frac{6e^t}{t} dt,$$

   find $f'(x)$.

7. Find the extrema of the function

$$f(x) = \int_1^{x^2} 12t - 3\, dt.$$

Answer Key

1. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \frac{\pi}{4}$

2. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 6\csc(x)\cot(x)\ dx = \frac{14}{\sqrt{3}} - 12$

3. $\int_1^2 \frac{6}{x} + \frac{1}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-3}\ dx = 6\ln(2) + \frac{3(4)^{\frac{1}{3}}}{5} - \frac{39}{80}$

4. $\int_0^3 f(x)\ dx = 8$

5. The total displacement of the object after 2 seconds is 0.4.

6. $f'(x) = \cos(x)\sin(\cos(x))(-\sin(x)) + \frac{6e^{\cos(x)}}{\cos(x)}(-\sin(x))$

7. f has minima at $x = \pm\frac{1}{2}$ and has a maximum at $x = 0$.

Solutions

1. Notice that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+(i/n)^2} &= \int_0^1 \frac{1}{1+x^2}\,dx \\
&= \arctan(x)\Big|_0^1 \\
&= \arctan(1) - \arctan(0) \\
&= \frac{\pi}{4}
\end{aligned}$$

2. We have

$$\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 6\csc(x)\cot(x)\ dx &= \tan(x) + 6\csc(x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \tan\left(\frac{\pi}{3}\right) + 6\csc\left(\frac{\pi}{3}\right) - \left(\tan\left(\frac{\pi}{6}\right) + 6\csc\left(\frac{\pi}{6}\right)\right) \\
&= \sqrt{3} + \frac{12}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}} + 12\right) \\
&= \frac{14}{\sqrt{3}} - 12
\end{aligned}$$

3. First, notice that we can rewrite

$$\int_1^2 \frac{1}{6x} + \frac{\sqrt[3]{x^2}}{2} - \frac{1}{2x^3}\,dx = \int_1^2 \frac{1}{6x} + \frac{1}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-3}\,dx$$

The power rule for integration implies that

$$\begin{aligned}
\int_1^2 \frac{6}{x} + \frac{1}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-3}\,dx &= \left(6\ln(x) + \frac{3}{10}x^{\frac{5}{3}} + \frac{1}{4}x^{-2}\right)\Big|_1^2 \\
&= \left(6\ln(2) + \frac{3}{10}2^{\frac{5}{3}} + \frac{1}{4}(2^{-2})\right) - \left(6\ln(1) + \frac{3}{10}1^{\frac{5}{3}} + \frac{1}{4}(1^{-2})\right) \\
&= 6\ln(2) + \frac{3(4)^{\frac{1}{3}}}{5} - \frac{39}{80}
\end{aligned}$$

4. We have

$$\begin{aligned}
\int_0^3 f(x)\ dx &= \int_0^1 4x^3 - 3x^2 dx + \int_1^3 2x dx \\
&= \left(x^4 - x^3\right)\Big|_0^1 + x^2\Big|_1^3 \\
&= (1^4 - 1^3) + (3^2 - 1^2) \\
&= 8
\end{aligned}$$

5. The total displacement after 2 seconds is given by

$$\begin{aligned}\int_0^2 -9.8t + 10dt &= \left(-4.9t^2 + 10t\right)\Big|_0^2 \\ &= -4.9(2)^2 + 10(2) \\ &= 0.4\end{aligned}$$

6. By the Fundamental Theorem of Calculus and the chain rule, we have that

$$f'(x) = \cos(x)\sin(\cos(x))(-\sin(x)) + \frac{6e^{\cos(x)}}{\cos(x)}(-\sin(x))$$

7. By the Fundamental Theorem of Calculus and the chain rule, we have that

$$f'(x) = \left(12x^2 - 3\right)(2x)$$

It follows that $f'(x) = 0$ when $x = \pm\frac{1}{2}$ or when $x = 0$. The first derivative test implies that f has minima at $x = \pm\frac{1}{2}$ and has a maximum at $x = 0$.