MAT126 Homework 2-3

Problems

1. If the acceleration of an object is $a(t) = -32$, the initial velocity of the object is $v(0) = 6$, and the initial position of the object is $s(0) = 10$, find the position of the object at time $t = 1$.

2. Estimate the area under the curve $f(x) = 4x - x^2$ between $x = 0$ and $x = 4$ using

   (a) 4 left hand rectangles

   (b) 4 right hand rectangles

3. Given the following table,

x	0	1	2	3	4	5	6
$f(x)$	1	2	5	9	17	26	37

   (a) estimate the area under $f(x)$ between $x = 0$ and $x = 6$ using a left hand Riemann sum with 6 rectangles of equal width,

   (b) estimate the area under $f(x)$ between $x = 0$ and $x = 6$ using a right hand Riemann sum with 6 rectangles of equal width.

   (c) estimate the area under $f(x)$ between $x = 0$ and $x = 6$ using a midpoint Riemann sum with 3 rectangles of equal width.

4. Write down the Riemann sum which approximates the area under $f(x) = x^2 - 1$ from $x = 0$ to $x = 2$ using n rectangles of equal width and right endpoints as heights of the rectangles.

5. The following sum can be interpreted as a right hand Riemann sum approximating the area under a curve $y = f(x)$ between $x = 8$ and $x = 10$ using n rectangles of equal width. Find the function $f(x)$.

$$\sum_{i=1}^{n} \sqrt{8 + \frac{2i}{n}} \frac{2}{n}.$$

Answer Key

1. $s(1) = 0$
2. (a) The left hand Riemann sum is equal to 10.
   (b) The right hand Riemann sum is equal to 10.
3. (a) The left hand Riemann sum is equal to 60.
   (b) The right hand Riemann sum is equal to 96.
   (c) The midpoint Riemann sum is equal to 74.
4. $\sum_{i=1}^{n} \frac{8i^2}{n^3} - \frac{2}{n}$
5. $f(x) = \sqrt{x}$

Solutions

1. The velocity of the object is antiderivative of the acceleration and is given by
$$v(t) = -32t + C.$$
Since the inital velocity of the object is $v(0) = 6$,
$$6 = -32(0) + C = C.$$
It follows that
$$v(t) = -32t + 6.$$
The position of the object is the antiderivative of the velocity and is given by
$$s(t) = -16t^2 + 6t + C.$$
Since the inital position of the object is $s(0) = 10$,
$$10 = -16(0)^2 + 6(0) + C = C.$$
It follows that
$$s(t) = -16t^2 + 6t + 10.$$
Plugging in $t = 1$ gives
$$s(1) = -16(1)^2 + 6(1) + 10 = 0.$$

2. The width of each rectangle is given by $\Delta x = \frac{4-0}{4} = 1$.
   (a) The left hand Riemann sum is given by
   $$(f(0) + f(1) + f(2) + f(3))\,\Delta x = (0 + 3 + 4 + 3)\,(1) \quad = 10$$
   (b) The right hand Riemann sum is given by
   $$(f(1) + f(2) + f(3) + f(4))\,\Delta x = (3 + 4 + 3 + 0)\,(1) \quad = 10$$

3. (a) The width of each rectangle is given by $\Delta x = \frac{6-0}{6} = 1$. The left hand Riemann sum is given by
   $$\begin{aligned}(f(0) + f(1) + f(2) + f(3) + f(4) + f(5))\,\Delta x &= (1 + 2 + 5 + 9 + 17 + 26)\,(1)\\ &= 60\end{aligned}$$
   (b) The width of each rectangle is given by $\Delta x = \frac{6-0}{6} = 1$. The right hand Riemann sum is given by
   $$\begin{aligned}(f(1) + f(2) + f(3) + f(4) + f(5) + f(6))\,\Delta x &= (2 + 5 + 9 + 17 + 26 + 37)\,(1)\\ &= 96\end{aligned}$$

(c) The width of each rectangle is given by $\Delta x = \frac{6-0}{3} = 2$. The midpoint Riemann sum is given by

$$(f(1) + f(3) + f(5)+)\, \Delta x = (2 + 9 + 26)\,(2) \quad = 74$$

4. The width of each rectangle is given by $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. Since we are using right endpoints, the height of the i^{th} is given by

$$f\left(\frac{2i}{n}\right) = \left(\frac{2i}{n}\right)^2 - 1$$

It follows that the right hand Riemann sum is given by

$$\sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \Delta x = \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^2 - 1\right) \frac{2}{n} \quad = \sum_{i=1}^{n} \frac{8i^2}{n^3} - \frac{2}{n}$$

5. Let $f(x) = \sqrt{x}$. Then n evenly spaced rectangles between $x = 8$ and $x = 10$ have width $\Delta x = \frac{10-8}{n} = \frac{2}{n}$. Using right endpoints, the height of the i^{th} rectangle is given by

$$f\left(8 + \frac{2i}{n}\right) = \sqrt{8 + \frac{2i}{n}}$$

and it follows that the right hand Riemann sum approximating the area under $f(x)$ is

$$\sum_{i=1}^{n} \sqrt{8 + i\frac{2}{n}}\,\frac{2}{n}$$