MAT125 Homework for Lectures 23

July 13, 2021

1 Problems

Compute the following limits:

1. $$\lim_{x\to 0} \frac{\sin^2 x}{x^2}.$$

2. Let $f(x) = (\sin(x)/x)^2$ be the function from question 1. Compute
$$\lim_{x\to 0} f'(x).$$

3. $$\lim_{t\to\infty} t\ln(1+1/t).$$

4. $$\lim_{x\to 0^+} x\ln(x).$$

5. $$\lim_{x\to\infty} x^{1/x}.$$

2 Answer Key

1. 1
2. 0
3. 1
4. 0
5. 1

3 Solution

1. We see that we cannot directly "plug in" 0 but we may use L'Hopital's rule. One application yields $2\sin(x)\cos(x)/2x$. The numerator is equal to $\sin(2x)$ (a trig identity). So the limit is of $\sin(2x)/2x$ which goes to 1 by a second application of L'Hopital's rule or the standard arguments from trigonometry.

2. By the quotient rule,

$$f'(x) = \frac{x^2\sin(2x) - 2x\sin^2(x)}{x^4} = \frac{x\sin(2x) - 2\sin^2(x)}{x^3}.$$

   Next, we see that we cannot directly plug in $x = 0$ so we apply L'Hopital's rule. Then

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{\sin(2x) + 2x\cos(2x) - 2\sin(2x)}{3x^2}.$$

   We apply L'Hopital's rule again:

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2\cos(2x) - 4x\sin(2x) - 2\cos(2x)}{6x} = \lim_{x\to 0} \frac{2\sin(2x)}{3} = 0.$$

3. Rewrite the function as $\ln(1+1/t)/(1/t)$. Applying L'Hopital's rule, we get that the limit equals

$$\lim_{t\to\infty} \frac{\frac{-1/t^2}{1+1/t}}{-1/t^2} = \lim_{t\to\infty} \frac{1}{1+1/t} = 1.$$

4. This is a $0 \times -\infty$ situation. Rewrite the function as $\frac{\ln(x)}{1/x}$. Applying L'Hopital's, this becomes $\frac{1/x}{-1/x^2} = -x$ The limit of this as $x \to 0^+$ is 0.

5. Let $y = x^{1/x}$; trivially, $y = e^{\ln(y)}$. But the exponential function is strictly increasing so $\lim_{x\to\infty} e^{\ln(y)} = \exp(\lim_{x\to 0}\ln(y))$. And $\ln(y) = \frac{\ln(x)}{x}$.

   Applying L'Hopital's rule to $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$. And $e^0 = 1$, the final answer.