MAT125 Homework for Lecture 20

July 12, 2021

1 Problems

Consider the function $f(x) = e^{1/(x^2+1)}$.

1. Find the 1st derivative of $f(x)$.
2. Find the critical points and values, written in (x, y)-form, of $f(x)$.
3. Find the 2nd derivative of $f(x)$.
4. Find the critical points of $f'(x)$ and label them as local max/min or inflection point.
5. Graph $f(x)$ and $f'(x)$ and label critical points and asymptotes.

2 Answer Key

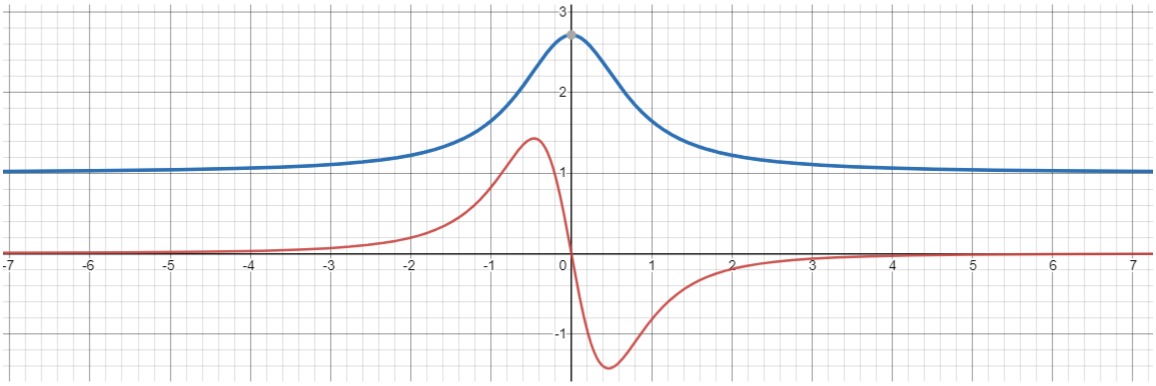
1.
$$f'(x) = -\frac{2x}{(x^2+1)^2}\exp(\frac{1}{x^2+1}).$$

2. $(0, e)$.

3.
$$f''(x) = \left(\frac{6x^4+8x^2-2}{(x^2+1)^4}\right)\exp(\frac{1}{x^2+1}).$$

4. $x = \pm\sqrt{(\sqrt{7}-2)/3}$. The negative root is the local max, the positive is the local min.

5. The blue curve is the graph of $f(x)$, the red is of $f'(x)$.



3 Solution

1. Write $f(x) = e^{g(x)}$ where $g(x) = (x^2+1)^{-1}$; $g'(x) = -2x(x^2+1)^{-2}$, and by the quotient rule:

$$g''(x) = \frac{-2(x^2+1)^2 - (-2x)2(x^2+1)2x}{(x^2+1)^4} = \frac{8x^2(x^2+1) - 2(x^2+1)^2}{(x^2+1)^4} = \frac{6x^4+4x^2-2}{(x^2+1)^4}.$$

   Then

$$f'(x) = g'(x)e^{g(x))} = -\frac{2x}{(x^2+1)^2}\exp(\frac{1}{x^2+1})$$

   by chain rule.

2. The global maximum of $f(x)$ can be found without the 1st derivative simply by noting that $1/(x^2+1)$ is largest when $x = 0$ and that e^x is an increasing function. The 1st derivative confirms there is only one critical point and the pair is $(0, e)$.

3. Using chain rule and product rule,

$$f''(x) = g''(x)e^{g(x)} + (g'(x))^2 e^{g(x)} = \left(\frac{6x^4+8x^2-2}{(x^2+1)^4}\right)\exp(\frac{1}{x^2+1}).$$

4. Set $f''(x) = 0$. This amounts to solving the quartic equation $3x^4+4x^2-1 = 0$. However, this reduces to a quadratic by letting $z = x^2$; so we can just use the quadratic formula for $3z^2+4z-1 = 0$. The solutions are

$$z = x^2 = \frac{-2 \pm \sqrt{7}}{3}.$$

   One of these is negative so there are no real solutions $x^2 =$ negative number. But $(\sqrt{7}-2)/3$ is positive and so this has two real square roots: $\pm\sqrt{(\sqrt{7}-2)/3}$. Note that plugging in the negative root into $f'(x)$ gives a positive value (just look at the signs) while the positive root gives a negative value.

5. See answer key for graphs.

   The graph of $f(x)$ is straightforward; there is one global max and the function is even and decays rapidly to the value e. So the horizontal asymptote is $y = e$.

   For graphing $f'(x)$, note that from above, $3z^2+4z-1$ is quadratic; well, secretly, it is quartic because $z = x^2$ but there are only 2 real roots. So this gives us the rough shape of $f'(x)$: $f'(x)$ is increasing before the negative root and after the positive root but decreasing between the roots. So we can see that $f'(x)$ is an odd function and in fact, the local extrema are also absolute extrema. As $x \to \pm\infty$, $-\frac{2x}{(x^2+1)^2} \to 0$ (bottom power is 4, top power is 1). So the horizontal asymptote for $f'(x)$ is $y = 0$.