MAT125 Homework for Lectures 18-19

July 5, 2021

1 Problems

1. A 25 ft ladder is leaning against a vertical wall. The bottom starts to slide away from the wall at 3 ft/s. How fast is the top sliding down when the top is 20 ft above the ground?

2. A conical tank has a height of 18 m and a radius of 6 m; it's positioned so that the nose of the cone is pointing groundward. It is filling with water at a rate of 14π m^3/min. How fast is the height of the water rising when the water is 10 m high?

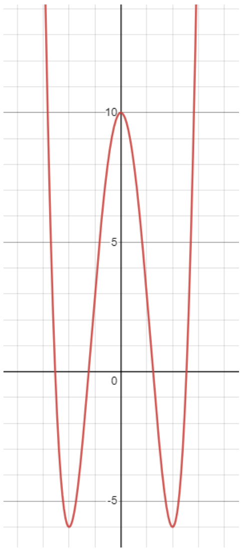
3. A rocket is launched vertically at 4 mi/s and you're standing 9 mi from the launch site. How fast is the angle of elevation changing after 3 seconds have passed?

4. Let $f(x) = x^4 - 8x^2 + 10$. Sketch a graph of the function, labeling the local extrema in (x, y)-form.

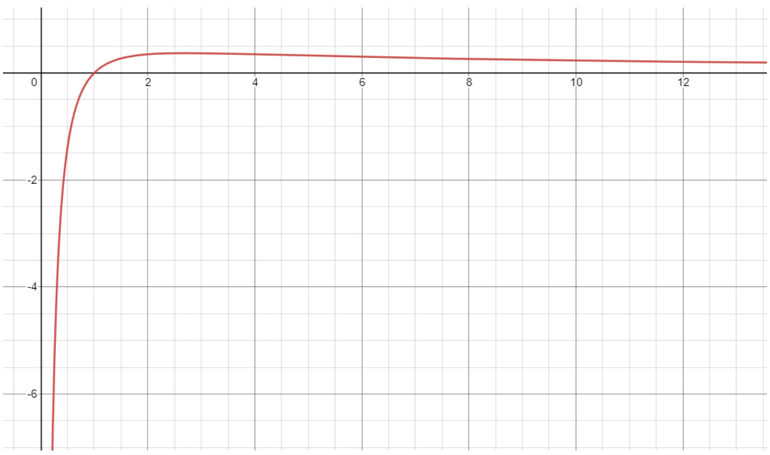
5. Let $g(x) = \frac{\ln(x)}{x}$. Sketch the graph of the function on its domain of definition, labeling the local extrema in (x, y)-form and also the x-intercept. Also, label the horizontal and vertical asymptotes.

2 Answer Key

1. $\frac{dy}{dt} = -\frac{9}{4}$ ft/s
2. $\frac{dh}{dt} = \frac{63}{50}$ m/min
3. $\frac{d\theta}{dt} = \frac{36}{225}$ rad/s
4. Local max: (0,10), absolute min: $(\pm 2, -6)$.



5. Absolute max: $(e, \frac{1}{e})$, x-intercept: $(1, 0)$. Horizontal asymptote: $y = 0$, vertical asymptote: $x = 0$.



3 Solution

1. Let x represent the horizontal distance between the foot of the ladder and the wall. Let y be the distance between the top of the ladder and the ground. Then, $x^2 + y^2 = 25^2$ is one relation between x and y. Differentiating, we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. On the other hand, when $y = 20$ ft, then $x^2 = 625 - 400 = 225 = 15^2$. So $x = 15$ ft. So plugging in, we have $2(15 \text{ ft})(3 \text{ ft/s}) + 2(20 \text{ ft})\frac{dy}{dt} = 0$. Then solve: $\frac{dy}{dt} = -\frac{9}{4}$ ft/s.

2. As the water rises, the radius and height of the cone of water changes but their ratio does not. The height is always $18/6 = 3$ times the radius. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Plugging in that $r = h/3$, we have $V = \frac{1}{27}\pi h^3$ and differentiating, $\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$. We're told that $\frac{dV}{dt} = 14\pi$ and we're asked to find $\frac{dh}{dt}$ when $h = 10$. Solving, we get $\frac{dh}{dt} = \frac{63}{50}$ m/min.

3. Let x represent the height of the rocket. The relationship between the angle of elevation θ and the height of the rocket is given by $\tan\theta = x/9$. So then differentiating, we have $\frac{dx}{dt} = 9\sec^2\theta\,\frac{d\theta}{dt}$. After 3 seconds, the rocket is 12 miles high. So $\tan\theta = 12/9$ and the hypotenuse of the triangle is 15 mi. So then, $\sec\theta = 15/9$ and hence we have the equation $4 = \frac{225}{9}\frac{d\theta}{dt}$. So $\frac{d\theta}{dt} = \frac{36}{225}$ rad/s.

4. $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$. So the critical points are $x = \pm 2, 0$. The 2nd derivative is $f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$ and its zeros are $\pm\frac{2}{\sqrt{3}}$. This tells us that the critical points are not inflection points but extrema and concavity tests show that $(0, 10)$ is concave down, hence a local max. The absolute min are $(\pm 2, -6)$. See picture above.

5. $\ln(x)$ is defined on $(0, \infty)$. By quotient rule, $g'(x) = (1 - \ln(x))/x^2$. So its one critical point is at $x = e$. Then $(e, \frac{1}{e})$ is its maximum. This can be checked by observing that $g'(x) > 0$ for $x < e$ and $g'(x) < 0$ for $x > e$.

   $(1, 0)$ is the only x-intercept since $\ln(x) < 0$ for $x < 1$ while $g(x) > 0$ for $x > 1$. $\lim_{x\to\infty} g(x) = 0$ gives $y = 0$ as a horizontal asympotote and $x = 0$ is the vertical asymptote.