1

Compute derivatives for the following functions:

1. $f(x) = \ln(x^2 + 2x + 1)$

2. $g(x) = x^2 \ln(1 + e^x)$

2

Describe the tangent line to the curve $y = e^{\arccos(x)-\pi}$, defined on the interval $[-1, 1]$, at the point $(\sqrt{2}/2, e^{-3\pi/4})$. Is the function decreasing or increasing near this point?

3

If the position $p(t)$ of a particle is given implicitly as a function of time by the following equation:

$$\ln(p(t)) = \ln(t^2 + 1) - \ln(t)$$

compute the instantaneous velocity of this particular at time $t = 1$. What does this tell you about the particle at this time?

4

Suppose the position of a particle at time t is given by the equation:

$$p(t) = \arctan(t^3)$$

Find the acceleration of the particle at time $t = 2$.

5

Find the equation of the tangent line to the curve $y = y(x)$ given by the equation $\cos(y) = \sqrt{2}x$ at the point $(1/2, \pi/4)$. In addition, determine a point at which the tangent line to this curve is vertical.

Answer Key

1. (i) $f'(x) = \frac{2}{x+1}$ (ii) $g'(x) = x^2e^x(1+e^x)^{-1} + 2x\ln(1+e^x)$.

2. The slope is $-\sqrt{2}e^{-3\pi/4}$, so the function is decreasing.

3. The instantaneous velocity is $p'(1) = 0$. In this case, the particle is finishing a period of deceleration prior to accelerating again.

4. The acceleration at time $t = 2$ is given by $p''(2) = -\frac{1524}{4225}$.

5. The equation of the tangent line at the given point is $y = -2x + (1 + \pi/4)$. The tangent line is vertical at the point $(1/\sqrt{2}, 0)$.

Solutions

1. Using the chain rule, we compute:

$$f'(x) = \frac{(x^2+2x+1)'}{x^2+2x+1} = \frac{2(x+1)}{(x+1)^2} = \frac{2}{x+1}$$

Using both the product rule and the chain rule, we compute:

$$g'(x) = x^2(\ln(1+e^x))' + 2x\ln(1+e^x) = \frac{x^2(1+e^x)'}{1+e^x} + 2x\ln(1+e^x) = \frac{x^2e^x}{1+e^x} + 2x\ln(1+e^x)$$

2. The slope of the tangent line to the curve $y = y(x)$ at the point $(\sqrt{2}/2, e^{-3\pi/4})$ is the derivative $\frac{dy}{dx}$ evaluated at the point $(\sqrt{2}/2, e^{-3\pi/4})$. Using the chain rule, we compute:

$$\frac{dy}{dx} = e^{\arccos(x)-\pi}(\arccos(x)-\pi)' = \frac{-e^{\arccos(x)-\pi}}{\sqrt{1-x^2}}$$

Evaluating at $(\sqrt{2}/2, e^{-3\pi/4})$, we know that $\arccos(\sqrt{2}/2) = \pi/4$ in this case, and a direct computation shows that:

$$\frac{dy}{dx}(\sqrt{2}/2) = \frac{-e^{\arccos(\sqrt{2}/2)-\pi}}{\sqrt{1-(\sqrt{2}/2)^2}} = \frac{-e^{-3\pi/4}}{\sqrt{1/2}} = -\sqrt{2}e^{-3\pi/4}$$

This slope is negative, so the function is decreasing at this point.

3. Using logarithm laws, we rewrite the defining equation for $p(t)$ as:

$$\ln(p(t)) = \ln(\frac{t^2+1}{t})$$

Using implicit differentiation, we obtain:

$$\frac{p'(t)}{p(t)} = \frac{t}{t^2+1}(\frac{t^2+1}{t})' = \frac{t}{t^2+1}\frac{t(t^2+1)' - (t^2+1)t'}{t^2} = \frac{t}{t^2+1}\frac{2t^2-(t^2+1)}{t^2} = \frac{t}{t^2+1}\frac{t^2-1}{t^2} = \frac{t^2-1}{t(t^2+1)}$$

The instantaneous velocity is the derivative of position, so we obtain:

$$p'(t) = p(t)\frac{t^2-1}{t(t^2+1)} \quad \Rightarrow \quad p'(1) = p(1)\frac{1^2-1}{1(1^2+1)} = p(1)\cdot 0 = 0$$

This means that at the instant $t = 1$, the particle is not moving. As can be seen from a graph, or by testing values, the particle is finishing a period of deceleration prior to time $t = 1$ and accelerates again after time $t = 1$.

4. First, we compute the velocity, using the chain rule

$$p'(t) = \frac{1}{1+(t^3)^2}(t^3)' = \frac{3t^2}{1+t^6}$$

Now, using the quotient rule, we compute the acceleration:

$$p''(t) = \frac{(1+t^6)(3t^2)' - (3t^2)(1+t^6)'}{(1+t^6)^2} = \frac{6t(1+t^6) - (3t^2)(6t^5)}{(1+t^6)^2} = \frac{6t + 6t^7 - 18t^7}{(1+t^6)^2} = \frac{6t - 12t^7}{(1+t^6)^2}$$

Hence, the acceleration at time $t = 2$ is:

$$p''(2) = \frac{6(2) - 12(2)^7}{(1+(2)^6)^2} = -\frac{1524}{4225}$$

5. Differentiating implicitly gives:

$$-sin(y)\frac{dy}{dx} = \sqrt{2} \quad \Rightarrow \quad \frac{dy}{dx} = -\sqrt{2}\csc(y)$$

Hence, at the point $(1/2, \pi/4)$, we have:

$$\frac{dy}{dx} = -\sqrt{2}\csc(\pi/4) = -\sqrt{2}\frac{2}{\sqrt{2}} = -2$$

Hence, the equation of the tangent line at the given point is:

$$y - \pi/4 = -2(x - 1/2) \quad \Rightarrow \quad y = -2x + (1 + \pi/4)$$

In order for the tangent line to be vertical at a given point, the limit of the derivative as x approaches that point needs to be $\pm\infty$. In particular, we see that this occurs when $\sin(y) = 0$, so for instance when $y = 0$. Hence, $1 = \cos(0) = \sqrt{2}x$, and thus $x = 1/\sqrt{2}$. Thus, the tangent line is vertical at the point $(1/\sqrt{2}, 0)$.