1

Compute the derivatives of the following functions:

1. $f(x) = e^x(e^x + x^2 + 1)$

2. $g(x) = e^x \cos(x)$

2

Suppose that the population $P(t)$ of trees in a forest, as a function of time (in years), is given by the equation:

$$P(t) = \frac{e^t}{t^2}$$

for all $t > 0$. Determine the time at which there is the lowest number of trees in the forest.

3

Determine all maxima of the function $f(t) = \cos(t)\sin(t)$.

4

Consider the functions:

$$f(x) = \frac{-a}{x} \quad \text{and} \quad g(x) = \frac{x^3}{3b}$$

Determine nonzero values of a and b for which the equation $f'(x) = g'(x)$ has no solutions.

5

If $h(x) = e^{-x}\cos(x)$, compute $h'(2\pi)$.

Answer Key

1. (i) $f'(x) = e^x(2e^x + x^2 + 2x) + 1$ (ii) $g'(x) = e^x(\cos(x) - \sin(x))$.

2. $t = 2$.

3. $t = \pi k/4$ for $k = \pm 1, \pm 5, \pm 9, \cdots$.

4. Any values of a and b such that $ab < 0$, for instance $a = 1$, $b = -1$.

5. $h'(2\pi) = -e^{-2\pi}$.

Solutions

1. Using the product rule, we see that:

$$f(x) = e^x(e^x + x^2 + 1)' + (e^x)'(e^x + x^2 + 1) = e^x(e^x + 2x) + e^x(e^x + x^2 + 1) = e^x(2e^x + x^2 + 2x) + 1$$

Again, using the product rule, we see that:

$$g'(x) = e^x(\cos(x))' + (e^x)'\cos(x) = -e^x\sin(x) + \cos(x)e^x = e^x(\cos(x) - \sin(x))$$

2. We first need to compute $P'(t)$ and then determine when $P'(t) = 0$. Using the quotient rule, we see that:

$$P'(t) = \frac{t^2(e^t)' - e^t(t^2)'}{(t^2)^2} = \frac{t^2e^t - 2te^t}{t^4} = \frac{e^t(t-2)}{t^3}$$

Hence, $P'(t) = 0$ precisely when $t = 2$. This is a minimum, as can be checked from a graph of $P(t)$ or by a table of values (note that $P(1) = e \approx 2.7$, $P(2) = e^2/4 \approx 1.8$, $P(3) = e^3/8 \approx 2.5$). Hence, year 2 was the time at which the population of trees in the forest was at a minimum.

3. First, we use the product rule to compute the derivative:

$$f'(t) = \cos(t)(\sin(t))' + (\cos(t))'\sin(t) = \cos^2(t) - \sin^2(t)$$

Now, $f'(t) = 0$ precisely when $\cos^2(t) = \sin^2(t)$ and hence precisely when $\cos(t) = \pm\sin(t)$. This occurs when $t = \pi k/4$, for any positive or negative odd integer k. By checking a graph, or by using a table of values, we see that the maxima correspond to the points when $k = \pm 1, \pm 5, \pm 9, \cdots$. In other words, when $k \equiv 1 \mod 4$. The mimima are when $k \equiv 3 \mod 4$.

4. We first use the quotient rule to compute:

$$f'(x) = \frac{x(-a)' - (-a)(x)'}{x^2} = \frac{a}{x^2}$$

Then, we use the power rule to compute:

$$g'(x) = \frac{3x^{3-2}}{3b} = \frac{x^2}{b}$$

Now, if $f'(x) = g'(x)$, then:

$$\frac{a}{x^2} = \frac{x^2}{b} \quad \Rightarrow x^4 = ab$$

There are no real number solutions x to the equation $x^4 = ab$ is $ab < 0$. Hence, we may take any values of a and b such that $ab < 0$, for instance $a = 1$, $b = -1$.

5. First, we use the quotient to compute:

$$(e^{-x})' = \left(\frac{1}{e^x}\right)' = \frac{e^x(1)' - 1(e^x)'}{e^{2x}} = \frac{-e^x}{e^{2x}} = \frac{-1}{e^x} = -e^{-x}$$

Now, we use the product rule and quotient rule to compute:

$$h'(x) = e^{-x}(\cos(x))' + (e^{-x})'\cos(x) = -e^{-x}\sin(x) - \cos(x)e^{-x} = -e^{-x}(\sin(x) + \cos(x))$$

Hence:

$$h'(2\pi) = -e^{-2\pi}(\sin(2\pi) + \cos(2\pi)) = -e^{-2\pi}(1 + 0) = -e^{-2\pi}$$