# 1

Compute the derivatives of the following functions:

1. $f(x) = x^4 + 4x^2 + 1$

2. $g(x) = (x^3 + 2x + 2)^2$

# 2

Let $f(x) = x^2 + 1$ and $g(x) = -x^3$. Consider the function:

$$h(x) = \begin{cases} f'(x) & x \geq 1 \\ ag'(x) & x < 1 \end{cases}$$

where $f'$ and $g'$ are the derivatives of $f$ and $g$, respectively. Determine the value of $a$ for which $h(x)$ is continuous.

# 3

Consider the function:

$$p(x) = \begin{cases} |x-3| & x \geq -1 \\ -x^2 & x < -1 \end{cases}$$

If any of the following limits do not exist, explain why. Otherwise, compute the limit:

1. $\lim_{x\to-\infty} p(x)$

2. $\lim_{x\to-1^+} p(x)$

3. $\lim_{x\to-1} p(x)$

# 4

Draw the graph of a function $f(x)$ that is discontinuous only at the points $x = -1$ and $x = 1$, and satisfies $\lim_{x\to-\infty} f(x) = \infty$.

# 5

Define a function piecewise (i.e., like in problems 2 and 3 above) that satisfies the conditions of the function $f(x)$ in problem 4. Prove that your example works.

## Answer Key

1. (i) $f'(x) = 4x^3 + 8x$ (ii) $g'(x) = 6x^5 + 16x^3 + 12x^2 + 8x + 8$.

2. $a = -2/3$.

3. (i) $-\infty$ (ii) $4$ (iii) Does not exist

4. Problems 4 and 5 can be answered simultaneously by considering a function defined piecewise as $-x$ on $(-\infty, -1)$, as $2$ on $[-1, 1]$, and as $3x$ on $(1, \infty)$. Many other possible functions would also suffice.

## Solutions

1. Using the power rule, we see that:

$$f'(x) = 4x^{4-1} + (4 \times 2)x^{2-1} + 0 = 4x^3 + 8x$$

For the derivative of $g(x)$, we first expand the square so that we can apply the power rule:

$$g(x) = (x^3 + 2x + 2)(x^3 + 2x + 2) = x^6 + 4x^4 + 4x^3 + 4x^2 + 8x + 4$$

Now, using the power rule, we see that:

$$g'(x) = 6x^{6-1} + (4 \times 4)x^{4-1} + (4 \times 3)x^{3-1} + (4 \times 2)x^{2-1} + 8 = 6x^5 + 16x^3 + 12x^2 + 8x + 8$$

2. Using the power rule, we compute $f'(x) = 2x$ and $g'(x) = -3x^2$. Since $f'$ and $g'$ are continuous, we see that in order for $h(x)$ to be continuous, it remains to require that $f'(1) = ag'(1)$. Indeed, this guarantees that $f'(1) = \lim_{x\to 1^+} h(x) = \lim_{x\to 1^-} h(x) = ag'(1)$, so that $h$ is continuous at $x = 1$. Now, $f'(1) = 2$ and $ag'(1) = -3a$, so $-3a = 2$ forces $a = -2/3$.

3. We have:

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} (-x^2) = -\infty$$

since $-x^2$ is concave down and goes down to $-\infty$ at both ends. Similarly, we compute:

$$\lim_{x\to-1^+} p(x) = \lim_{x\to-1^+} |x-3| = |-1-3| = |-4| = 4$$

Since $|x - 3| = p(x)$ for all $x \geq -1$ ("from the right hand side of $-1$"). Lastly, the limit $\lim_{x\to-1} p(x)$ does not exist, since $\lim_{x\to-1^-} p(x) \neq \lim_{x\to-1^+} p(x)$. Indeed:

$$\lim_{x\to-1^-} p(x) = \lim_{x\to-1^-} (-x^2) = -1 \neq 4$$

4, 5. For problems 4 and 5, many possible functions would do the trick. We consider the following function, defined piecewise as:

$$f(x) = \begin{cases} -x & x < -1 \\ 2 & -1 \leq x \leq 1 \\ 3x & x \geq 1 \end{cases}$$

This function is discontinuous at the points $x = -1$ and $x = 1$, since:

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} (-x) = 1 \neq 2 = \lim_{x\to-1^+} 2 = \lim_{x\to-1^+} f(x)$$

and

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 2 \neq 3 = \lim_{x\to1^+} 3x = \lim_{x\to1^+} f(x)$$

Finally, we verify that $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(-x) = \infty$, as required.