Lecture #16
MAT 123

Compound Interest, e, and the natural logarithm

$\log_B X = A$ means $B^A = X$

* logs help scale things especially in science

ex: $\log_2 \frac{1}{512} = x$

$2^x = \frac{1}{512}$

$2^{-9} = \frac{1}{512}$

$\boxed{x = -9}$

powers of 2:
2
4
8
16
32
64
128
256
$\boxed{512}$ => $2^9 = 512$

ex: $\log_4 2^{12.6} = x$

$4^x = 2^{12.6}$

$(2^2)^x = 2^{12.6}$

$\frac{\cancel{2}x}{\cancel{2}} = \frac{12.6}{2}$

$\boxed{x = 6.3}$

* we were able to make both sides have the same base → this let us set the exponents equal to each other.

ex: $\log_6 x = 3$

$6^3 = x$

$\boxed{x = 216}$

ex: $\log_b 625 = \frac{4}{3}$

$\left(b^{\frac{4}{3}}\right)^{\frac{3}{4}} = (625)^{\frac{3}{4}}$

$b = 625^{\frac{3}{4}} = \left(\sqrt[4]{625}\right)^3$

$\boxed{b = 125}$

* $\frac{4}{3} \cdot \frac{3}{4} = 1$

by raising the the reciprocal power we are able to "cancel" out the power

$$\boxed{x^{\frac{a}{b}} = y \iff x = y^{\frac{b}{a}}}$$

general case

logs and exponentials are inverses of each other

Remember this means we reflect our graph over the line $y=x$

If $f(x) = \log_3 x$. find $f^{-1}(x)$

$$y = \log_3 x$$
$$x = \log_3 y$$
$$3^x = y$$

$$f^{-1}(x) = 3^x$$

so if $(9,2)$ is a point on $y = \log_3 x$
then $(2,9)$ must be a point on $y = 3^x$
(b/c of the inverse property)

If $f(x) = \log_4(2x-1)$ find $f^{-1}(x)$.

$$y = \log_4(2x-1)$$
$$x = \log_4(2y-1)$$
$$4^x = 2y - \cancel{1}$$
$$+1 \qquad \cancel{+1}$$

$$\frac{4^x+1}{2} = \frac{\cancel{2}y}{\cancel{2}}$$
$$y = \frac{4^x+1}{2}$$

$$f^{-1}(x) = \frac{4^x+1}{2}$$

What if $f(x) = 3^{x+1}$. find $f^{-1}(x)$

$$y = 3^{x+1}$$
$$x = 3^{y+1}$$
$$\log_3 x = y + \cancel{1}$$
$$-1$$

$$y = \log_3(x) - 1$$

$$f^{-1}(x) = \log_3(x) - 1$$

OR $\quad x = 3^{y+1}$ take the log of both sides.

$$\frac{\log x}{\log 3} = (y+1)\frac{\cancel{\log 3}}{\cancel{\log 3}}$$

$$y = \frac{\log x}{\log 3} - 1$$

$$f^{-1}(x) = \frac{\log x}{\log 3} - 1$$

"change of base rule:"

$$\boxed{\log_B X = \frac{\log_c X}{\log_c B}}$$

→ hence why:

$$\log_3 X - 1 = \frac{\log X}{\log 3} - 1$$

(from the last problem)

ex: $\log_4 11 = \dfrac{\log 11}{\log 4}$

Compound Interest

$$F = P(1+R)^T$$

T → time

R → interest rate

F ← future amount

P ← principal amount or initial amount or present amount

<u>example</u>: Suppose you deposited \$1,000 at 6% interest, for 5 years. What will it be worth?

$F = P(1+R)^T$

$F = ?$
$P = 1000$
$R = 0.06$ → *notice you have to turn the rate into a decimal to use the formula.
$T = 5$

$F = 1{,}000(1.06)^5$

$F = 1338.2256$

\$1,338.23

$$F = P\left(1+\frac{R}{N}\right)^{NT}$$

N = # of times you are paid interest per year. "interest on interest"

$F = 1000\left(1+\frac{.06}{2}\right)^{2(5)}$

$F = 1000(1.03)^{10}$ ← compounded twice a year, or semiannually.

$F = 1343.92$

$1343.92 > 1338.23$

3% twice a year is better than 6% once a year what about monthly?

$$F = 1000\left(1 + \frac{.06}{12}\right)^{12(5)}$$

$F = 1348.85$ ← even better!

notice the more frequently compounded the more \$ you make!

what about continuously compounding?

$$F = P\left(1 + \frac{R}{\infty}\right)^{\infty}$$

$$\boxed{F = Pe^{RT}}$$

continuous compounding

$\left(1 + \frac{1}{N}\right)^N \rightarrow e = 2.71828$

also used for bacteria growth

ex: you invest \$1000 at 6% compounded $\underbrace{\text{continuously}}_{\text{means all the time}}$. for 5 years. How much is it worth?

$$F = 1000e^{(.06)(5)} = \$1349.86$$

Natural Logarithm ⟶ $\ln X$

$\log_B X = A$ means $B^A = X$

$\ln X = A$ means $e^A = X$

$\log_e$

Natural Log Rules:

$\ln(1) = 0 \Rightarrow e^0 = 1$

$\ln(e) = 1$

$\ln(e^X) = X$

$\ln(AB) = \ln(A) + \ln(B)$

$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$

$\ln A^B = B \cdot \ln A$

ex: If $f(x) = \ln(2x-5)$ find $f^{-1}(x)$

$$\boxed{f^{-1}(x) = \frac{e^x+5}{2}}$$

$$y = \ln(2x-5)$$

$$e^{x} = e^{\ln(2y-5)}$$

$$e^x = 2y - \cancel{5}$$
$$+5 \qquad \cancel{+5}$$

$$\frac{e^x+5}{2} = \frac{\cancel{2}y}{\cancel{2}}$$

$$y = \frac{e^x+5}{2}$$

ex: If $f(x) = e^{1+6x}$ find $f^{-1}(x)$

$$\boxed{f^{-1}(x) = \frac{\ln(x)-1}{6}}$$

$$y = e^{1+6x}$$

$$\ln x = \ln e^{1+6y}$$

$$\ln(x) = \cancel{1} + 6y$$
$$-1 \qquad \cancel{-1}$$

$$\frac{\ln(x)-1}{6} = \frac{\cancel{6}y}{\cancel{6}}$$

$$y = \frac{\ln(x)-1}{6}$$

ex: If $f(x) = e^{4x-3} + 2$ Find $f^{-1}(x)$

$$y = e^{4x-3} + 2$$

$$x = e^{4y-3} + \cancel{2}$$
$$-2 \qquad \cancel{-2}$$

$$\ln x-2 = \ln e^{4y-3}$$

$$\ln(x-2) = 4y - \cancel{3}$$
$$+3 \qquad \cancel{+3}$$

$$\frac{\ln(x-2)+3}{4} = \frac{\cancel{4}y}{\cancel{4}}$$

$$\frac{\ln(x-2)+3}{4} = y$$

$$\boxed{f^{-1}(x) = \frac{\ln(x-2)+3}{4}}$$

ex:

If $f(x) = 2\ln\left(\frac{x}{5}\right) + 1$ find $f^{-1}(x)$

$$y = 2\ln\left(\frac{x}{5}\right) + 1$$

$$x = 2\ln\left(\frac{y}{5}\right) + \cancel{1}$$
$$-1 \qquad\qquad\qquad \cancel{-1}$$

$$\frac{x-1}{2} = \frac{\cancel{2}\ln\left(\frac{y}{5}\right)}{\cancel{2}}$$

$$e^{\frac{x-1}{2}} = e^{\ln\left(\frac{y}{5}\right)}$$

$$5 \cdot e^{\left(\frac{x-1}{2}\right)} = \frac{y}{\cancel{5}} \cdot \cancel{5}$$

$$y = 5e^{\left(\frac{x-1}{2}\right)}$$

$$f^{-1}(x) = 5e^{\left(\frac{x-1}{2}\right)}$$