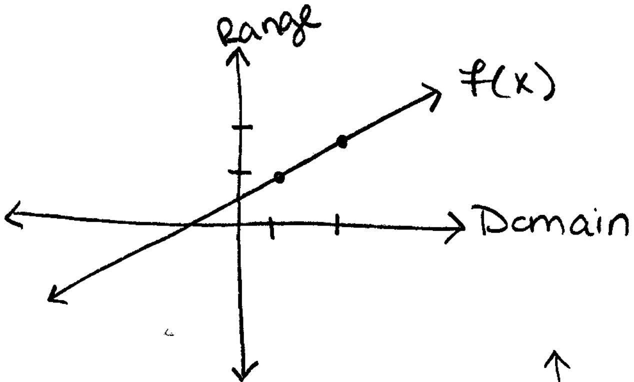
Lecture #7
MAT 123

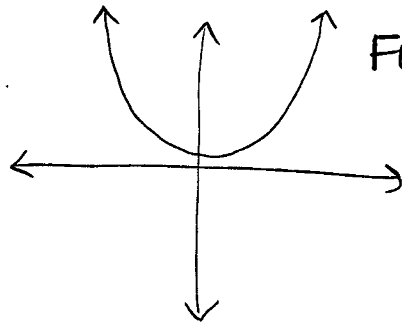
Composition of Functions

Recall: A function is an operation that maps an input to exactly one output.

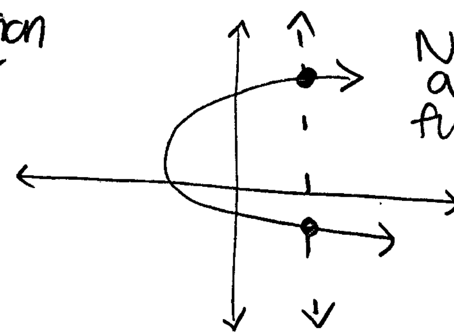


vertical line test: No vertical line can intersect the line more than once

if a graph passes this test it is a function.



Function ✓



Not a function

Composition of Functions: Basic Rules

$f(x) = x^2 + 3 \qquad g(x) = \frac{1}{x}$

$(f+g)(x) = ? = x^2 + 3 + \frac{1}{x}$

$(f-g)(x) = ? = x^2 + 3 - \frac{1}{x}$

$(fg)(x) = (x^2+3)(\frac{1}{x})$

$(\frac{f}{g})(x) = \frac{x^2+3}{\frac{1}{x}} = (x^2+3)(x)$

ex: $f(x) = x^2 + 3 \quad g(x) = \frac{1}{x}$

$f(g(x))$ "f of g of x"
also written as

$(f \circ g)(x) = (g(x))^2 + 3$

$= \left(\frac{1}{x}\right)^2 + 3$

"go to f(x) and whenever you see x replace it with g(x)"

$(g \circ f)(x) = \frac{1}{f(x)} = \frac{1}{x^2+3}$

"go to g(x) and whenever you see x replace it with f(x)"

* we work from right to left!

ex: $f(x) = x^3 - x + 2 \quad g(x) = 2x + 1$

① $f(2) = ?$

$= (2)^3 - 2 + 2$

$= \boxed{8}$

② $g(4) = ?$

$= 2(4) + 1 = \boxed{9}$

③ $f(g(4)) = ?$

$g(4) = 2(4) + 1 = 9$

$f(g(4)) = f(9)$

$f(9) = 9^3 - 9 + 2 = \boxed{722}$

ex: $f(x) = x^2 + 2x + 1$

$f(x+h) = ?$

"everytime you see x in f(x) replace it with (x+h)"

$f(x+h) = (x+h)^2 + 2(x+h) + 1$

$= (x+h)(x+h) + 2x + 2h + 1$

$= \boxed{x^2 + 2xh + h^2 + 2x + 2h + 1}$

note: $(x+h)^2 = x^2 + 2xh + h^2$

$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

ex: $f(x) = 3x^2 - x + 4$

① $f(x+h) = ?$
② $f(x+h) - f(x) = ?$

Don't forget to distribute the negative

① $f(x+h) = 3(x+h)^2 - (x+h) + 4$
$= 3(x^2 + 2xh + h^2) - x - h + 4$
$= \boxed{3x^2 + 6xh + 3h^2 - x - h + 4}$

② $f(x+h) - f(x) = 3x^2 + 6xh + 3h^2 - x - h + 4 - (3x^2 - x + 4)$
$= \cancel{3x^2} + 6xh + 3h^2 \cancel{-x} - h + \cancel{4} \cancel{-3x^2} \cancel{+x} \cancel{-4}$
$= \boxed{6xh + 3h^2 - h}$

* notice → all the terms without an "h" cancel out.

ex: $f(x) = x^3 + 4x - 1$
① $f(x+h) = ?$
② $f(x+h) - f(x) = ?$
③ $\dfrac{f(x+h) - f(x)}{h} = ?$

① $f(x+h) = (x+h)^3 + 4(x+h) + 1$
$= \boxed{x^3 + 3x^2h + 3xh^2 + h^3 + 4x + 4h - 1}$

* Don't forget the ( )!

② $f(x+h) - f(x) = x^3 + 3x^2h + 3xh^2 + h^3 + 4x + 4h - 1 - (x^3 + 4x - 1)$
$= \cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{4x} + 4h \cancel{-1} \cancel{-x^3} \cancel{-4x} \cancel{+1}$
$= \boxed{3x^2h + 3xh^2 + h^3 + 4h}$

③ $\dfrac{f(x+h) - f(x)}{h} = \underbrace{\dfrac{3x^2\cancel{h} + 3xh^{\cancel{2}\,1} + h^{\cancel{3}\,2} + 4\cancel{h}}{\cancel{h}}}_{\text{factor an } h \text{ out of each term}} = \boxed{3x^2 + 3xh + h^2 + 4}$

this is called the "Difference Quotient"

↳ in calculus we use this in the definition of the derivative.

example: $f(x) = \frac{1}{x} + 2$

What is $f(x+h) - f(x) = ?$

$$f(x+h) = \frac{1}{x+h} + 2$$

$$f(x+h) - f(x) = \frac{1}{x+h} + 2 - \left(\frac{1}{x} + 2\right)$$

$$= \frac{1}{x+h} + \cancel{2} - \frac{1}{x} - \cancel{2}$$

common denorminator.

$$= \frac{1}{x+h} - \frac{1}{x} = \frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}$$

$$= \frac{x - (x+h)}{x(x+h)} = \frac{x - x + h}{x(x+h)} = \boxed{\frac{-h}{x(x+h)}}$$