Homework

For problems 1 and 2, determine:

- intervals where f is increasing and decreasing
- all local maxima and minima
- intervals where f is concave up and concave down
- all inflection points
- and sketch the graph of the function, marking the critical points and inflection points

Problems:

1. $f(x) = 1 - 3x^2 + x^3$
2. $f(x) = x^4 + 4x$
3. Draw the graph of a function that is both concave up and decreasing for all real numbers.
4. Draw the graph of a function that is both concave down and decreasing for all real numbers.
5. Draw the graph of a function that is concave down for $x < 0$, concave up for $x > 0$, increasing on $(-\infty, -1) \cup (1, \infty)$, and decreasing on $(-1, 1)$. Label the inflection point on your graph. (It must have an inflection point since the concavity changes from being concave up to concave down).

Solutions

1. $f(x) = 1 - 3x^2 + x^3$

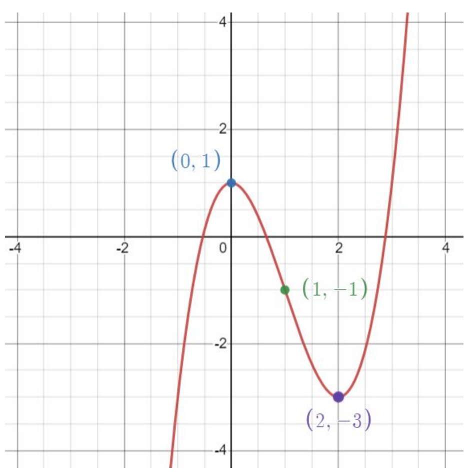
   Solution: To find where f is increasing or decreasing, we must find when it has positive or negative slope. Hence, we need to compute it's derivative. Using power rule, we get $f'(x) = 0 - 3 \cdot 2x + 3x^2 = -6x + 3x^2$. Since the derivative is continuous, we can find the intervals on which it is increasing or decreasing by first finding where $f'(x) = 0$. Setting equal to 0, we get $0 = -6x + 3x^2 = 3x(-2 + x)$. And so $f'(x) = 0$ at $x = 0, 2$. Picking a point less than 0, say -1, we see $f'(-1) = -6(-1) + 3(-1)^2 = 6 + 3 = 9$. So for $x < 0$, $f'(x) < 0$. $f'(1) = -6 + 3 = -3$ so for $0 < x < 2$, $f'(x) > 0$. Finally, $f'(3) = -18 + 27 = 9$ so for $x > 2$, $f'(x) > 0$. That is, f is increasing on $(-\infty, 0) \cup (2, \infty)$ and f is decreasing on $(0, 2)$.

   At $x = 0$, $f(x)$ goes from increasing to decreasing, hence it is a local max. At $x = 2$, $f(x)$ goes from decreasing to increasing, hence it is a local min. Since these are the only critical points, there can be no other local maxima/minima. One can also use the second derivative to show f has a max at $x = 0$ and a min at $x = 2$.

   To find where f is concave up or concave down, we must find the intervals where f'' is positive or negative. $f''(x) = \frac{d}{dx}[f'(x)] = -6 + 6x$. Factoring out a 6, we get $f''(x) = 6(x - 1)$. So for $x < 1$, $x - 1 < 0$ so f is concave down on $(-\infty, 1)$. And for $x > 1$, $x - 1 > 0$ so f is concave up.

   $f''(1) = 0$ and the concavity changes from being concave up to concave down so f has an inflection point at $x = 1$ and this is the only inflection point.

   In order to graph the function, we must also know the y-values corresponding to the critical points and inflection points. Plugging into our function, we get $f(0) = 1$, $f(2) = -3$, $f(1) = -1$.



2. $f(x) = x^4 + 4x$.

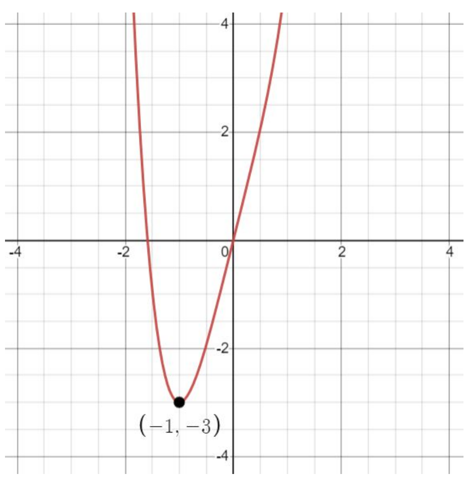
Solution: To find where f is increasing or decreasing, we must find when it has positive or negative slope. Hence, we need to compute it's derivative. Using power rule, we get $f'(x) = 4x^3 + 4$. Since the derivative is continuous, we can find the intervals on which it is increasing or decreasing by first finding where $f'(x) = 0$. Setting equal to 0, we get $0 = 4x^3 + 4 = 4(x^3 + 1)$. And so $f'(x) = 0$ when $x^3 + 1 = 0$ which occurs at $x = -1$. Picking a point less than -1, say -2, we see $f'(-2) = 4(-2)^3 + 4 = -32 + 4 = -28 < 0$ So for $x < 1$, $f'(x) < 0$. $f'(0) = 4(0)^3 + 4 = 4 > 0$ so for $x > -1$, $f'(x) > 0$. That is, f is decreasing on $(-\infty, -1)$ and f is increasing on $(-1, \infty)$.

At $x = 1$, f goes from decreasing to increasing, hence it is a local min. Since this is the only critical point, there can be no other local maxima/minima. One can also use the second derivative to show f has a min at $x = -1$.

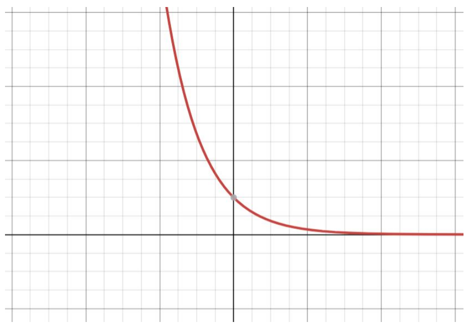
To find where f is concave up or concave down, we must find the intervals where f'' is positive or negative. $f''(x) = \frac{d}{dx}[f'(x)] = 12x^2$. This is a relatively simple function and we can see that f'' is zero only at $x = 0$ and positive for all other values of x. Hence, it is concave up on $(-\infty, 0) \cup (0, \infty)$.

$f''(0) = 0$, however, the concavity doesn't change at this point – it is positive on both sides of 0. Since this is the only point where $f''(x) = 0$, f has no inflection points.

In order to graph the function, we must also know the y-value corresponding to the critical point. Plugging into our function, we get $f(-1) = (-1)^4 + 4(-1) = -3$.

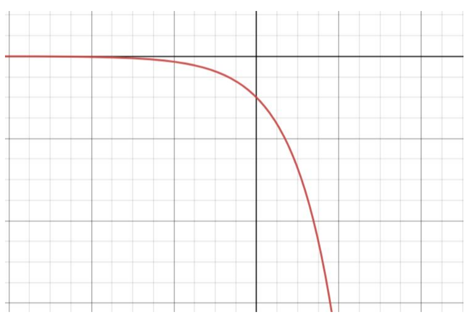


3. Draw the graph of a function that is both concave up and decreasing for all real numbers.



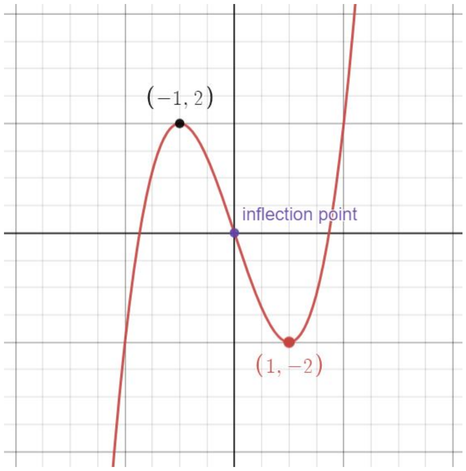
The above curve is concave up since the slope is always increasing. It starts with a large negative slope and this large negative slope is always increasing approaching a 0 (or horizontal) slope. One can verify this using calculus as this is the graph of $y = e^{-x}$.

4. Draw the graph of a function that is both concave down and decreasing for all real numbers.



The above curve is decreasing for all real numbers, as can be seen visually. It is also concave down as the slope continually gets more and more negative (or steeper and steeper in a downwards direction). One can also verify this using calculus as this is the graph of $y = -e^{x}$.

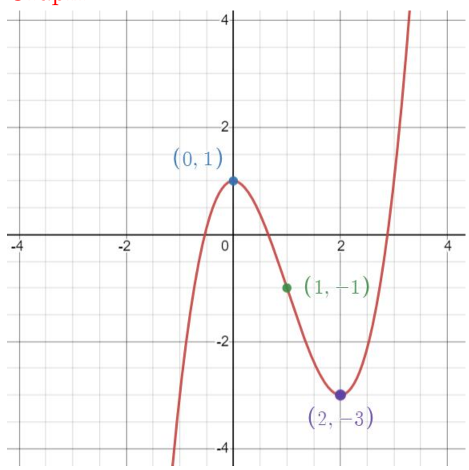
5. Draw the graph of a function that is concave down for $x < 0$, concave up for $x > 0$, increasing on $(-\infty, -1) \cup (1, \infty)$, and decreasing on $(-1, 1)$. Label the inflection point on your graph. (It must have an inflection point since the concavity changes from being concave up to concave down).



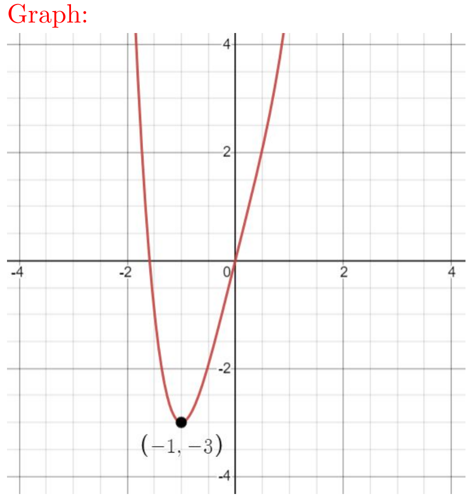
The above curve is concave down on $(-\infty, 0)$ because on this interval, the slope is decreasing from being large and positive to large and negative. At $x = 0$ the slope is at its most negative and then increases to be large and positive. This happens on $(0, \infty)$ so the curve is concave up on $(0, \infty)$. The inflection point occurs at $x = 0$. We've marked the maxima/minima and can see visually that the function is increasing on $(-\infty, -1) \cup (1, \infty)$ and is decreasing on $(-1, 1)$. This can all be verified using calculus as this is the graph of $y = x^3 - 3x$.

Answer Key

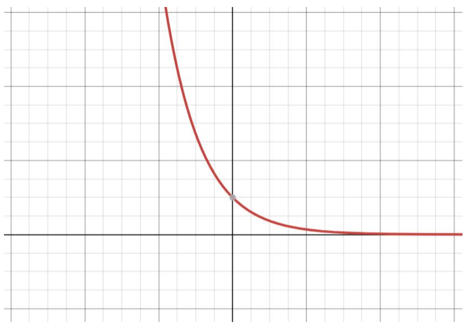
1. $f(x) = 1 - 3x^2 + x^3$
   - f increasing on $(-\infty, 0) \cup (2, \infty)$, decreasing on $(0, 2)$.
   - f has a maximum at $(0, 1)$ and a minimum at $(2, -3)$
   - f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.
   - f has an inflection point at $(1, -1)$.
   - Graph:



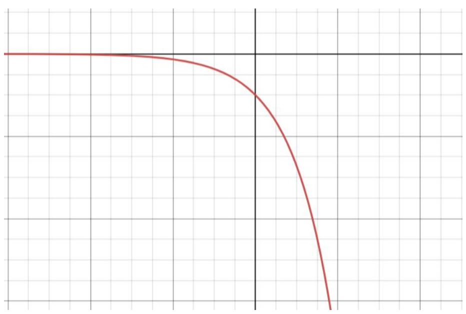
2. $f(x) = x^4 + 4x$
   - f decreasing on $(-\infty, -1)$, increasing on $(-1, \infty)$.
   - f has a minimum at $(-1, -3)$
   - f is concave up on $(\infty, 0) \cup (0, \infty)$.
   - f has no inflection points.
   - Graph:



3. Draw the graph of a function that is both concave up and decreasing for all real numbers.



4. Draw the graph of a function that is both concave down and decreasing for all real numbers.



5. Draw the graph of a function that is concave down for $x < 0$, concave up for $x > 0$, increasing on $(-\infty, -1) \cup (1, \infty)$, and decreasing on $(-1, 1)$. Label the inflection point on your graph. (It must have an inflection point since the concavity changes from being concave up to concave down).