Homework

For problems 1 - 3, determine:

- intervals where f is increasing and decreasing
- all critical points and local maxima/minima

Problems:

1. $f(x) = 3x^4 - 6x^3 + 5$
2. $f(x) = x^3 - 3x^2 - 24x + 7$
3. $f(x) = 3x(x-2)^{2/3}$
4. A Roman army in the time of emperor Marcus Aurelius utilized catapults to fire on the enemy. If the vertical position of the projectile was given by $y(t) = -5t^2 + 40t + 10$, at what time did the projectile reach a maximum height? What was the maximum height of the projectile?
5. Draw the curve for the function in problem 1, labeling all critical points.
6. Draw the curve for the function in problem 2, labeling all critical points.
7. Draw the curve for the function in problem 3, labeling all critical points.

Solutions

1. $f(x) = 3x^4 - 6x^3 + 5$

   Solution: In order to find critical points and where f is increasing/decreasing, we must first compute f'. $f'(x) = 12x^3 - 18x^2 = 6x^2(2x-3)$. Since f' is defined for all values of x, the critical points occur when $f'(x) = 0$. Setting equal to 0, we get $0 = 6x^2(2x-3)$. Therefore, $f'(x)$ equals 0 only when $x^2 = 0$ or when $2x - 3 = 0$. Hence we have critical points at $x = 0$ and $x = 3/2$. Since these are the only zeroes of f', f' is either all positive or all negative on each of the intervals $(-\infty, 0)$, $(0, 3/2)$, and $(3/2, \infty)$. To find which it is, it suffices to evaluate f' at a single point in each interval. Since $f'(-1) = 12(-1)^3 - 18(-1)^2 = -12 - 18 = -30 < 0$, f is decreasing on $(-\infty, 0)$. Since $f'(1) = 12(1)^3 - 18(1)^2 = 12 - 18 = -6 < 0$, f is also decreasing on $(0, 3/2)$. Since $f'(2) = 12(2)^3 - 18(2)^2 = 96 - 72 = 24 > 0$, f is increasing on $(3/2, \infty)$.

   Since f is decreasing on both sides of 0, 0 is neither a local max nor local min. Since f goes from decreasing to increasing at $3/2$, f has a local min at $3/2$.

2. $f(x) = x^3 - 3x^2 - 24x + 7$

   Solution: In order to find critical points and where f is increasing/decreasing, we must first compute f'. $f'(x) = 3x^2 - 6x - 24$. Since we want to find the zeroes of f' in order to find the critical points, it helps to factor f'. Doing so, we get $f'(x) = 3(x-4)(x+2)$. Hence, f' equals 0 at $x = 4$ and $x = -2$. Since f' is defined for all x, these are the only critical points. We continue as in the previous problem and evaluate f' at a single point in each of the intervals $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$. $f'(-3) = 3(-3-4)(-3+2) = 3(-7)(-1) = 21 > 0$. So f is increasing on $(-\infty, -2)$. $f'(0) = 3(0-4)(0+2) = 3(-4)(2) = -24 < 0$. So f is decreasing on $(-2, 4)$. $f'(5) = 3(5-4)(5+2) = 3(1)(7) = 21 > 0$. So f is increasing on $(4, \infty)$. We have critical points only at $x = 4$, $x = -2$. At $x = -2$, f goes from increasing to decreasing so f has a local max at $x = -2$. At $x = 4$, f goes from decreasing to increasing so we have a local min at $x = 4$.

3. $f(x) = 3x(x-2)^{2/3}$

   Solution: In order to find critical points and where f is increasing/decreasing, we must first compute f'. Using the product rule, we get

$$
\begin{aligned}
f'(x) &= 3(x-2)^{2/3} + 3x\frac{2}{3}(x-2)^{2/3-1} = 3(x-2)^{2/3} + 2x(x-2)^{-1/3} \\
&= (x-2)^{-1/3}(3(x-2) + 2x) = (x-2)^{-1/3}(5x-6)
\end{aligned}
$$

   Hence, f' is undefined at $x = 2$ and 0 at $x = 6/5$. These are the two only critical points for f. We continue as in the previous problem and evaluate f' at a single point in each of the intervals $(-\infty, 6/5)$, $(6/5, 2)$, and $(2, \infty)$.

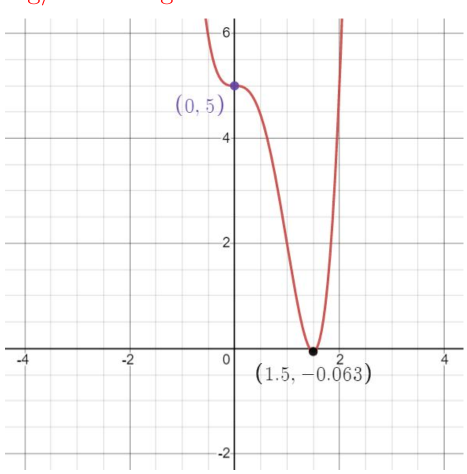
$f'(0) = (0-2)^{-1/3}(5(0)-6) = (-2)^{-1/3}(-6) > 0$. So f is increasing on $(-\infty, 6/5)$. $f'(7/5) = ((7/5)-2)^{-1/3}(5(7/5)-6) = (-3/5)^{-1/3}(1) < 0$. So f is decreasing on $(6/5, 2)$. $f'(3) = ((3)-2)^{-1/3}(5(3)-6) = 1^{-1/3}(9) = 9 > 0$. So f is increasing on $(2, \infty)$. Since $f(2)$ is defined and f goes from decreasing to increasing at $x = 2$, f has a local min at $x = 2$. Since f goes from increasing to decreasing at $x = 6/5$, f has a local max at $x = 6/5$.

4. A Roman army in the time of emperor Marcus Aurelius utilized catapults to fire on the enemy. If the vertical position of the projectile was given by $y(t) = -5t^2 + 40t + 10$, at what time did the projectile reach a maximum height? What was the maximum height of the projectile?

   Solution: The maximum height of a function occurs either at a critical point. So we must first take the derivative of our height function. $y'(t) = -10t + 40$. This is defined for all values of t and is 0 when $t = 4$. Since y' is positive for $t < 4$ and negative for $t > 4$, $y(t)$ has a max at time $t = 4$. Plugging this into the original function, will give us the value of the maximum height. $y(4) = -5(4)^2 + 40(4) + 10 = 90$. So the maximum height of the projectile is 90 units.

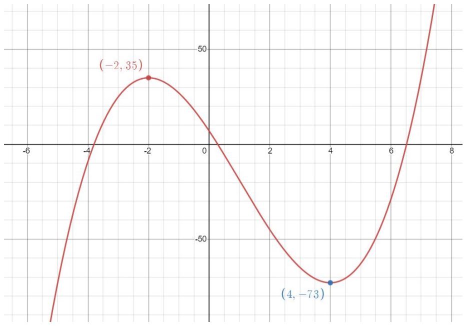
5. Draw the curve for the function in problem 1, labeling all critical points.

   This can be done by first finding the y values associated with the critical points and drawing a smooth curve between them which is increasing/decreasing on the correct intervals.



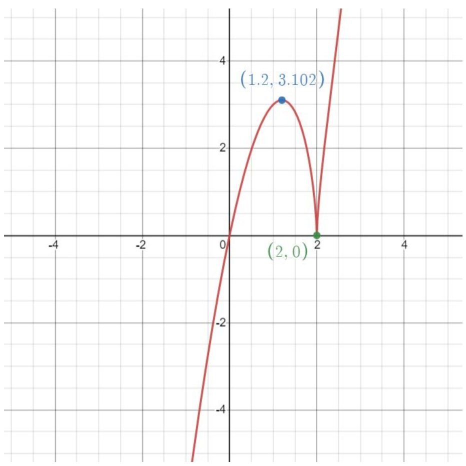
6. Draw the curve for the function in problem 2, labeling all critical points.

   This can be done by first finding the y values associated with the critical points and drawing a smooth curve between them which is increasing/decreasing on the correct intervals.



7. Draw the curve for the function in problem 3, labeling all critical points.

This can be done by first finding the y values associated with the critical points and drawing a smooth curve between them which is increasing/decreasing on the correct intervals. The only place the curve should not be smooth is at $x = 2$ where the derivative does not exist.



Answer Key

1. $f(x) = 3x^4 - 6x^3 + 5$

   Decreasing on $(-\infty, 0) \cup (0, 3/2)$. Increasing on $(3/2, \infty)$. Critical points at $x = 0, 3/2$. A local min at $x = 3/2$.

2. $f(x) = x^3 - 3x^2 - 24x + 7$

   Decreasing on $(-2, 4)$. Increasing on $(-\infty, -2) \cup (4, \infty)$. Critical points at $x = -2, 4$. A local min at $x = 4$, a local max at $x = -2$.

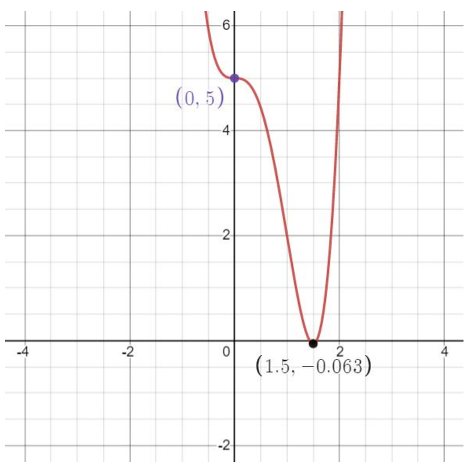
3. $f(x) = 3x(x-2)^{2/3}$

   Decreasing on $(6/5, 2)$. Increasing on $(-\infty, 6/5) \cup (2, \infty)$. Critical points at $x = 6/5, 2$. Local max at $x = 6/5$. Local min at $x = 2$.

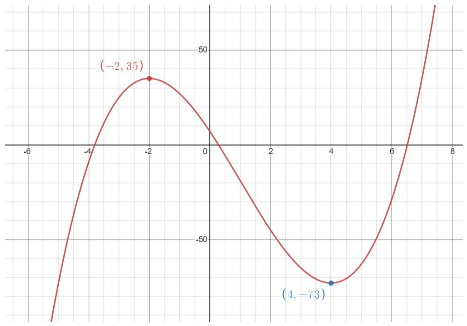
4. A Roman army in the time of emperor Marcus Aurelius utilized catapults to fire on the enemy. If the vertical position of the projectile was given by $y(t) = -5t^2 + 40t + 10$, at what time did the projectile reach a maximum height? What was the maximum height of the projectile?

   Maximum height occurs at $t = 4$. The maximum height itself is $y(4) = 90$.

5. Draw the curve for the function in problem 1, labeling all critical points.



6. Draw the curve for the function in problem 2, labeling all critical points.



7. Draw the curve for the function in problem 3, labeling all critical points.