MAT122 Homework 18-20

Problems

1. Use the product rule to compute the derivatives of the following functions:

   (a) $f(x) = \left(12x^2 - x\right)\left(x^3 - 7x^2 + 9\right)$

   (b) $f(x) = \left(1 + \sqrt{x^5}\right)\left(\frac{1}{x} - 5\sqrt[3]{x}\right)$

2. Use the quotient rule to compute the derivatives of the following functions:

   (a) $f(x) = \frac{6x^3}{2 - x}$

   (b) $f(x) = \frac{3x + x^4}{2x^2 + 1}$

3. Use the chain rule to compute the derivatives of the following functions:

   (a) $f(x) = \left(6x^2 + 7x\right)^4$

   (b) $f(x) = \sqrt[4]{1 - 8x^2}$

4. If $f(3) = 10$, $f'(3) = 7$, $g(3) = 3$, $g'(3) = -4$, find

   (a) $(fg)'(3)$

   (b) $(\frac{f}{g})'(3)$

   (c) $(f \circ g)'(3)$

5. Find the equation of the tangent line to

$$f(x) = x^6\sqrt{5x^2 - 1}$$

   at $x = 1$.

6. Find the second derivative of the function

$$f(x) = \frac{1}{(x^2 + 1)^2}.$$

Answer Key

1. (a) $f'(x) = 60x^4 - 340x^3 + 21x^2 + 216x - 9$

   (b) $f'(x) = \frac{3}{2}x^{\frac{1}{2}} - x^{-2} - \frac{5}{3}x^{\frac{-2}{3}} - \frac{85}{6}x^{\frac{11}{6}}$

2. (a) $\frac{12x^2(3-x)}{(2-x)^2}$

   (b) $\frac{4x^5 + 4x^3 - 6x^2 + 3}{(2x^2+1)^2}$

3. (a) $f'(x) = 4(6x^2 + 7x)^3(12x + 7)$

   (b) $f'(x) = \frac{-4x}{(1-8x^2)^{\frac{3}{4}}}$

4. (a) $(fg)'(3) = 58.$

   (b) $\left(\frac{f}{g}\right)'(3) = \frac{82}{9}.$

   (c) $(f \circ g)'(3) = -40.$

5. $y = \frac{29}{2}x - \frac{25}{2}$

6. $f''(x) = \frac{20x^2 - 4}{(x^2+1)^4}$

Solutions

1. (a) By the product rule,

$$\begin{aligned} f'(x) &= (24x-1)(x^3-7x^2+9)+(12x^2-x)(3x^2-14x) \\ &= 24x^4-168x^3+216x-x^3+7x^2-9+36x^4-168x^3-3x^3+14x^2 \\ &= 60x^4-340x^3+21x^2+216x-9 \end{aligned}$$

   (b) Notice that

$$f(x)=\left(1+x^{\frac{5}{2}}\right)\left(x^{-1}-5x^{\frac{1}{3}}\right).$$

   It follows that

$$\begin{aligned} f'(x) &= \frac{5}{2}x^{\frac{3}{2}}\left(x^{-1}-5x^{\frac{1}{3}}\right)+\left(1+x^{\frac{5}{2}}\right)\left(-x^{-2}-\frac{5}{3}x^{\frac{-2}{3}}\right) \\ &= \frac{5}{2}x^{\frac{1}{2}}-\frac{25}{2}x^{\frac{11}{6}}-x^{-2}-\frac{5}{3}x^{\frac{-2}{3}}-x^{\frac{1}{2}}-\frac{5}{3}x^{\frac{11}{6}} \\ &= \frac{3}{2}x^{\frac{1}{2}}-x^{-2}-\frac{5}{3}x^{\frac{-2}{3}}-\frac{85}{6}x^{\frac{11}{6}} \end{aligned}$$

2. (a) By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{18x^2(2-x)-6x^3(-1)}{(2-x)^2} \\ &= \frac{36x^2-12x^3}{(2-x)^2} \\ &= \frac{12x^2(3-x)}{(2-x)^2} \end{aligned}$$

   (b) By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(3+4x^3)(2x^2+1)-(3x+x^4)(4x)}{(2x^2+1)^2} \\ &= \frac{6x^2+3+8x^5+4x^3-12x^2-4x^5}{(2x^2+1)^2} \\ &= \frac{4x^5+4x^3-6x^2+3}{(2x^2+1)^2} \end{aligned}$$

3. (a) By the chain rule, $f'(x)=4(6x^2+7x)^3(12x+7)$.

   (b) Notice that

$$f(x)=(1-8x^2)^{\frac{1}{4}}.$$

   By the chain rule,

$$f'(x)=\frac{1}{4}(1-8x^2)^{\frac{-3}{4}}(-16x)=\frac{-4x}{(1-8x^2)^{\frac{3}{4}}}.$$

4. (a) By the product rule,

$$(fg)'(3) = f'(3)g(3) + f(3)g'(3) = 10(7) + 3(-4) = 70 - 12 = 58.$$

(b) By the quotient rule,

$$\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{g(3)^2} = \frac{10(7) - 3(-4)}{3^2} = \frac{82}{9}.$$

(c) By the chain rule,

$$(f \circ g)'(3) = f(g(3))g'(3) = f(3) * (-4) = 10(-4) = -40.$$

5. By the product rule and chain rule,

$$f'(x) = 6x^5\left(\sqrt{5x^2-1}\right) + x^6(5x^2-1)^{\frac{-1}{2}}(10x)$$

It follows that $f'(1) = \frac{29}{2}$. Since $f(1) = 2$, the point-slope formula implies that the equation of the tangent line is

$$y - 2 = \frac{29}{2}(x-1).$$

It follows that the equation of the tangent line is

$$y = \frac{29}{2}x - \frac{25}{2}.$$

6. Notice that

$$f(x) = (x^2+1)^{-2}.$$

By the chain rule, the first derivative is

$$f'(x) = -2(x^2+1)^{-3}(2x) = \frac{-4x}{(x^2+1)^3}$$

The quotient rule and chain rule imply that

$$\begin{aligned} f''(x) &= \frac{-4(x^2+1)^3 + 12x(x^2+1)^2(2x)}{(x^2+1)^6} \\ &= \frac{(x^2+1)^2(-4(x^2+1) + 24x^2)}{(x^2+1)^6} \\ &= \frac{20x^2-4}{(x^2+1)^4} \end{aligned}$$