MAT122 Homework 13-17

Problems

1. Find the domain of the following functions:
   (a) $f(x) = \sqrt{x^2+1} - \sqrt[3]{1-x}$
   (b) $f(x) = \dfrac{\sqrt{x-1}}{x^2-3x-4}$.
2. If $f(0) = 2$ and $g(2) = 0$,
   (a) Find $(f \circ g)(2)$.
   (b) Find $(g \circ f)(0)$.
3. We want to build a rectangular fence in a field whose length is twice the width and we have 80 feet of fencing material. If we use all the fencing material what would the dimensions of the fence be?
4. Compute the following limits, if they exist:
   (a) $\lim_{x\to 0} \dfrac{x^2+6}{x^2-3}$
   (b) $\lim_{x\to 3} \dfrac{x^2-16x+3}{9-x^2}$
   (c) $\lim_{x\to 1} \dfrac{10-9x-x^2}{3x^2+4x-7}$
   (d) $\lim_{x\to \infty} \dfrac{10-9x-x^2}{3x^2+4x-7}$
5. Let $f(x) = x^2 + 6x + 5$.
   (a) Find $\dfrac{f(x+h)-f(x)}{h}$.
   (b) Use the definition of the derivative to find $f'(x)$.
6. Find the derivative of $f(x) = x^3 - \dfrac{1}{x^6} + \dfrac{1}{\sqrt[5]{x^2}}$
7. Find the equation of the tangent line to
   $$f(x) = 2x - x^2$$
   at $x = 2$.

Answer Key

1. (a) $(-\infty, \infty)$

   (b) $[1, 4) \cup (4, \infty)$

2. (a) $(f \circ g)(2) = 2$.

   (b) $(g \circ f)(0) = 0$.

3. The width of the fence is $\dfrac{40}{3}$ feet and the length of the fence is $\dfrac{80}{3}$ feet.

4. (a) $\displaystyle\lim_{x\to 0} \frac{x^2+6}{x^2-3} = -2$

   (b) $\displaystyle\lim_{x\to 3^+} \frac{x^2-16x+3}{9-x^2}$ does not exist.

   (c) $\displaystyle\lim_{x\to 1} \frac{10-9x-x^2}{3x^2+4x-7} = -\frac{11}{10}$

   (d) $\displaystyle\lim_{x\to \infty} \frac{10-9x-x^2}{3x^2+4x-7} = -\frac{1}{3}$

5. (a) $\dfrac{f(x+h)-f(x)}{h} = 2x+h+6$

   (b) $f'(x) = 2x+6$

6. $f'(x) = 3x^2 + 6x^{-7} - \dfrac{2}{5}x^{-\frac{7}{5}}$

7. $y = -2x + 4$

Solutions

1. (a) Since $x^2 + 1 \geq 0$ for all x, the domain of $\sqrt{x^2+1}$ is $(-\infty, infty)$. Since we can take cube roots of negative numbers, the domain of $\sqrt[3]{1-x}$ is $(-\infty, \infty)$. It follows that the domain of $f(x)$ is $(-\infty, \infty)$.

   (b) Since $x - 1 \geq 0$ for $x \geq 1$, the domain of $\sqrt{x-1}$ is $[1, \infty)$. Now notice that the denominator can be factored as
   $$x^2 - 3x - 4 = (x-4)(x+1).$$
   It follows that the denominator is equal to 0 when $x = -1$ and $x = 4$. It follows that the domain of $f(x)$ is
   $$[1, 4) \cup (4, \infty).$$

2. (a) $(f \circ g)(2) = f(g(2)) = f(0) = 2$.

   (b) $(g \circ f)(0) = g(f(0)) = g(2) = 0$.

3. If the width of the fence is x feet, then the equation for the perimeter of the rectangular fence gives
   $$2x + 2(2x) = 80.$$
   It follows that $6x = 80$ and we conclude that $x = \frac{80}{6} = \frac{40}{3}$. So the width of the fence is $\frac{40}{3}$ feet and the length of the fence is $\frac{80}{3}$ feet.

4. (a) $\lim_{x \to 0} \frac{x^2+6}{x^2-3} = \frac{0^2+6}{0^2-3} = -2$

   (b) Notice that
   $$\lim_{x \to 3^-} \frac{x^2 - 16x + 3}{9 - x^2} = -\infty,$$
   and
   $$\lim_{x \to 3^+} \frac{x^2 - 16x + 3}{9 - x^2} = \infty.$$
   Since the limit from the left does not equal the limit from the right, the limit does not exist.

   (c) Plugging in $x = 1$, we see that
   $$\frac{10 - 9(1) - 1^2}{3(1)^2 + 4(1) - 7} = \frac{0}{0}.$$
   However, notice that
   $$\frac{x^2 - 16x + 3}{9 - x^2} = \frac{(10+x)(1-x)}{(3x+7)(x-1)} = \frac{-(10+x)}{3x+7}.$$
   It follows that
   $$\lim_{x \to 1} \frac{10 - 9x - x^2}{3x^2 + 4x - 7} = \lim_{x \to 1} \frac{-(10+x)}{3x+7} = \frac{-(10+1)}{3(1)+7} = -\frac{11}{10}$$

(d) Since the highest power on the numerator and the denominator are the same, the limt is the ratio of the coefficients in front of the highest powers. It follows that

$$\lim_{x\to\infty} \frac{10 - 9x - x^2}{3x^2 + 4x - 7} = -\frac{1}{3}.$$

5. (a) We have

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + 6(x+h) + 5 - (x^2 + 6x + 5)}{h} \\
&= \frac{x^2 + 2xh + h^2 + 6x + 6h + 5 - x^2 - 6x - 5}{h} \\
&= \frac{2xh + h^2 + 6h}{h} \\
&= \frac{h(2x + h + 6)}{h} \\
&= 2x + h + 6.
\end{aligned}$$

(b) By definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} 2x + h + 6 = 2x + 0 + 6 = 2x + 6.$$

6. Notice that

$$f(x) = x^3 - x^{-6} + x^{-\frac{2}{5}}.$$

It follows from the power rule that

$$f'(x) = 3x^2 + 6x^{-7} - \frac{2}{5}x^{-\frac{7}{5}}.$$

7. By the power rule,

$$f'(x) = 2 - 2x.$$

It follows that $f'(2) = -2$. Since $f(2) = 0$, the point-slope formula says that equation for the line with slope -2 passing through the point $(2, 0)$ has the equation

$$y - 0 = -2(x - 2).$$

It follows that the equation of the tangent line is

$$y = -2x + 4.$$