Lecture 29

# Equations Reducible to Quadratic

## Applications of quadratic equations

In this lecture we will learn how to apply our knowledge about quadratic equations to other problems.

We will discuss

- Polynomial equations
- Biquadratic equations
- Rational equations
- Word problems leading to quadratic equations



## Polynomial Equations

**Example 1.** Solve the equation $x^3 - 3x^2 - 4x = 0$.

**Solution.** This is a **polynomial equation**, since $x^3 - 4x^2 - 3x$ is a polynomial.

To solve the equation, we factor LHS:

$x^3 - 4x^2 - 3x = 0 \iff x(x^2 - 4x - 3) = 0$.

The product of two factors, $x$ and $x^2 - 4x - 3$, equals $0$

if and only if $x = 0$ or $x^2 - 4x - 3 = 0$.

By this, the first root is $x_1 = 0$. To find other roots,

we have to solve the quadratic equation $x^2 - 4x - 3 = 0$.

$$x^2 - 4x - 3 = 0 \iff x = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 + 12}}{2}$$

$$= \frac{4 \pm \sqrt{28}}{2} = \frac{4 \pm 2\sqrt{7}}{2} = 2 \pm \sqrt{7}.$$

Therefore, the equation has three roots: $x_1 = 0$, $x_2 = 2 + \sqrt{7}$, $x_3 = 2 - \sqrt{7}$.

## Biquadratic equations

**Example 2.** Solve the equation $x^4 + 2x^2 - 3 = 0$.

**Solution.** This equation is called **biquadratic**.

It is solved by the **substitution** $t = x^2$. Observe that $t \geq 0$.

$$x^4 + 2x^2 - 3 = 0 \iff t^2 + 2t - 3 = 0 \iff (t-1)(t+3) = 0$$
$$\iff t = 1 \text{ or } t = -3.$$

Since $t \geq 0$, we reject the negative root $t = -3$.

By this, the only solution is given by $t = 1$, that is $x^2 = 1$. So $x = \pm 1$.

**Answer.** $x = \pm 1$



## Rational equations

**Example 3.** Solve the equation $\dfrac{1}{x} + \dfrac{2}{x+1} = 1$.

**Solution.** This equation is called **rational**, since it contains rational expressions.

To solve the equation, we bring RHS to $0$:

$$\frac{1}{x} + \frac{2}{x+1} = 1 \iff \frac{1}{x} + \frac{2}{x+1} - 1 = 0.$$

Bring all terms to the **common denominator**:

$$\frac{x+1}{x(x+1)} + \frac{2x}{x(x+1)} - \frac{x(x+1)}{x(x+1)} = 0$$

Combine the terms in a single fraction:

$$\frac{x+1+2x-x(x+1)}{x(x+1)} = 0$$ and simplify

$$\frac{-x^2+2x+1}{x(x+1)} = 0$$

## Rational equations

We have got that the original equation is equivalent to the following equation:

$$\frac{-x^2+2x+1}{x(x+1)} = 0\,.$$

When is a fraction equal to $0$?

Only if its **numerator** equals $0$ and the denominator is **not** equal to $0$ (since one can't divide by $0$).

Therefore,

$$\frac{-x^2+2x+1}{x(x+1)} = 0 \iff -x^2+2x+1=0 \text{ and } x \neq 0\,,\ x \neq -1\,.$$

Let us solve the quadratic equation:

$$-x^2+2x+1=0 \iff x^2-2x-1=0$$

$$\iff x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm 2\sqrt{2}}{2} = \boxed{1\pm\sqrt{2}}$$

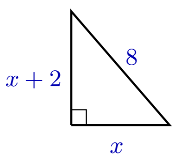
We accept both roots, since none of them is $0$ or $-1$.



## Word problems

**Problem 1.** The hypotenuse of a right triangle is $8$ cm long.
One leg is $2$ cm shorter than the other. Find the lengths of the legs of the triangle.

**Solution.**




Let $x$ cm be the length of the shorter leg.
Then the other leg has the length of $x+2$ cm.
The hypotenuse is $8$ cm.

By the **Pythagorean** theorem, $x^2 + (x+2)^2 = 8^2$.

To find $x$, we have to solve this quadratic equation.

## Word problems

To solve the equation, we have bring it to the **standard** form.

$$x^2 + (x+2)^2 = 8^2 \iff x^2 + x^2 + 4x + 4 = 64 \iff 2x^2 + 4x - 60 = 0$$
$$\iff x^2 + 2x - 30 = 0\,.$$

The equation is in the standard form now, and we can use the **quadratic formula**:

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-30)}}{2} = \frac{-2 \pm \sqrt{124}}{2} = \frac{-2 \pm 2\sqrt{31}}{2} = -1 \pm \sqrt{31}\,.$$

We have got two solutions, $x_1 = -1 + \sqrt{31}$ and $x_2 = -1 - \sqrt{31}$.

One of the solutions, $x_2 = -1 - \sqrt{31}$, is **negative**, and should be rejected, since $x$, being the length of a side in a triangle, is positive.

Therefore, one leg is $-1 + \sqrt{31}$ cm long, the other leg is $-1 + \sqrt{31} + 2 = 1 + \sqrt{31}$ cm long.

**Answer.** The lengths of the legs are $-1 + \sqrt{31}$ cm and $1 + \sqrt{31}$ cm.

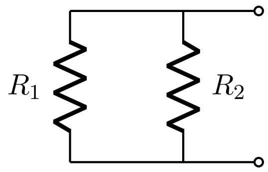


## Word problems

**Problem 2.** Two parallel resistors provide the total resistance of 2 Ohms. Find the value of each resistor if one of them is 3 Ohms more than the other. Use the law for parallel resistors:

$$\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2}.$$

**Solution.**




Given: $R_{\text{total}} = 2$, $R_2 = R_1 + 3$.
Plug these into the given equation:

$$\frac{1}{2} = \frac{1}{R_1} + \frac{1}{R_1 + 3}.$$

To find $R_1$, we have to solve this **rational** equation.

## Word problems

$\frac{1}{2} = \frac{1}{R_1} + \frac{1}{R_1 + 3} \iff \frac{1}{R_1} + \frac{1}{R_1 + 3} - \frac{1}{2} = 0$

Bring all terms to the common denominator:

$\frac{2(R_1+3)}{2R_1(R_1+3)} + \frac{2R_1}{2R_1(R_1+3)} - \frac{R_1(R_1+3)}{2R_1(R_1+3)} = 0$ Combine the terms in a single fraction:

$\frac{2(R_1+3) + 2R_1 - R_1(R_1+3)}{2R_1(R_1+3)} = 0$ Simplify:

$\frac{-R_1^2 + R_1 + 6}{2R_1(R_1+3)} = 0 \iff -R_1^2 + R_1 + 6 = 0 \iff R_1^2 - R_1 - 6 = 0$

$\iff (R_1 - 3)(R_1 + 2) = 0 \iff R_1 = 3$ or $R_1 = -2$.

We reject the negative root $R_1 = -2$ since a negative resistance makes no sense.

So $R_1 = 3$ Ohms and $R_2 = R_1 + 3 = 3 + 3 = 6$ Ohms.



## Summary

In this lecture, we have learned

- ☑ how to solve **polynomial** equations reducible to quadratic ones
- ☑ how to solve **biquadratic** equations
- ☑ how to solve **rational** equations
- ☑ how to solve word problems leading to quadratic equations