Lecture 26

Quadratic Equations

Quadratic polynomials

A **quadratic polynomial** is a polynomial of degree two.

It can be written in the standard form $ax^2 + bx + c$,
where x is a variable, a, b, c are constants (numbers) and $a \neq 0$.

The constants a, b, c are called the **coefficients** of the polynomial.

Example 1 (quadratic polynomials).

$-3x^2 + x - \frac{4}{5}$ ($a = -3,\ b = 1,\ c = -\frac{4}{5}$)

x^2 ($a = 1,\ b = c = 0$)

$\frac{x^2}{7} - 5x + \sqrt{2}$ ($a = \frac{1}{7},\ b = -5,\ c = \sqrt{2}$)

$4x(x+1) - x$ (this is a quadratic polynomial which is not written in the standard form. Its standard form is $4x^2 + 3x$, where $a = 4, b = 3, c = 0$)

Quadratic polynomials

Example 2 (polynomials, but not quadratic)
$x^3 - 2x + 1$ (this is a polynomial of degree 3, not 2)
$3x - 2$ (this is a polynomial of degree 1, not 2)
Example 3 (not polynomials)
$x^2 + x^{\frac{1}{2}} + 1$, $x - \frac{1}{x}$ are not polynomials

A quadratic polynomial $ax^2 + bx + c$ is called sometimes a **quadratic trinomial**.
A trinomial consists of three terms.

Quadratic polynomials of type $ax^2 + bx$ or $ax^2 + c$
are called **quadratic binomials**. A binomial consists of two terms.

Quadratic polynomials of type ax^2 are called **quadratic monomials**.
A monomial consists of one term.

Quadratic polynomials (together with polynomials of degree 1 and 0) are the **simplest** polynomials.

Due to their simplicity, they are among the most important algebraic objects.

Quadratic equations and their roots

A **quadratic equation** is an equation that can be written in the standard form $ax^2 + bx + c = 0$, where x is an unknown, a, b, c are constants and $a \neq 0$.

Examples. $-x^2 + 3x + 5 = 0$ is quadratic equation in standard form with
$a = -1, b = 3, c = 5$.

$x + 1 = 2x(3 - 4x)$ is a quadratic equation, but not in standard form.
We obtain its standard form as follows:

$$x + 1 = 2x(3 - 4x) \iff x + 1 = 6x - 8x^2 \iff 8x^2 - 5x + 1 = 0.$$

To **solve** an equation means to find **all** values of the unknown which turn the equation into a numerical identity.

The values of x that turn the equation $ax^2 + bx + c = 0$ into a numerical identity are called the **roots** or **solutions** of the equation.

Also, they are called the **roots** of the polynomial $ax^2 + bx + c$.

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How to solve a binomial quadratic equation

Example 1. Solve the equation $x^2 - 3 = 0$.

Solution. Alternative 1.

$$x^2 - 3 = 0 \iff x^2 = 3 \iff \sqrt{x^2} = \sqrt{3} \iff |x| = \sqrt{3}$$
$$\iff x = \sqrt{3} \text{ or } x = -\sqrt{3}.$$

One can shorten the answer: $x = \pm\sqrt{3}$.

Alternative 2. Let us write 3 as $(\sqrt{3})^2$ and use the **difference of squares** formula:

$$x^2 - 3 = 0 \iff x^2 - (\sqrt{3})^2 = 0 \iff (x - \sqrt{3})(x + \sqrt{3}) = 0.$$

The product of two terms, $(x - \sqrt{3})$ and $(x + \sqrt{3})$, equals 0 **if and only if** either one term equals 0, or the other term equals 0:

$$(x - \sqrt{3})(x + \sqrt{3}) = 0 \iff x - \sqrt{3} = 0 \text{ or } x + \sqrt{3} = 0$$
$$\iff x = \sqrt{3} \text{ or } x = -\sqrt{3}$$

Answer. $x = \pm\sqrt{3}$.

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Solution in simplest radical form

Example 2. Solve the equation $3x^2 - 5 = 0$. Give the answer in simplest radical form.

Solution.

$$3x^2 - 5 = 0 \iff 3x^2 = 5 \iff x^2 = \frac{5}{3} \iff x = \pm\sqrt{\frac{5}{3}}.$$

To write the number $\sqrt{\frac{5}{3}}$ in the **simplest radical form**,

we have to get rid of the radical in the denominator:

$$\sqrt{\frac{5}{3}} = \frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}\,\sqrt{3}}{\sqrt{3}\,\sqrt{3}} = \frac{\sqrt{15}}{3}.$$

Therefore, the solution is $x = \pm\sqrt{\frac{5}{3}} = \pm\frac{\sqrt{15}}{3}$.

Answer. $x = \pm\frac{\sqrt{15}}{3}$.

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Quadratic equations with no roots

Example 3. Solve the equation $x^2 + 4 = 0$.

Solution. $x^2 + 4 = 0 \iff x^2 = -4$.

We know that the square of any real numbers is **non-negative** (positive or zero).

Therefore, the equation has **no** real solutions.

Example 4. Solve the equation $x(2 - 3x) = (x + 1)^2$.

Solution. The equation is **not** in the standard form. Let us bring it to this form.

$$x(2 - 3x) = (x + 1)^2 \iff 2x - 3x^2 = x^2 + 2x + 1$$
$$\iff -4x^2 = 1 \iff x^2 = -\frac{1}{4}.$$

The square of a real number can't be negative, therefore, the equation has **no** real solutions.

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Solving binomial equations by factoring

Example 5. Solve the equation $-3x^2 + 4x = 0$.

Solution. By **factoring**, we get

$-3x^2 + 4x = 0 \iff x(-3x + 4) = 0$.

The product of two unknown numbers, x and $-3x + 4$ equals zero.

This may happen **if and only if** either one number equals 0 , or the other number equals 0 :

$x(-3x + 4) = 0 \iff x = 0$ or $-3x + 4 = 0 \iff x = 0$ or $x = \frac{4}{3}$.

Answer. $x = 0$ or $x = \frac{4}{3}$

Don't lose roots!

Example 6. Solve the equation $x(x - 1) = x$.

Solution. Rewrite the equation to bring it to the standard form:

$x(x - 1) = x \iff x^2 - x = x \iff x^2 - 2x = 0$.

Solve this binomial equation by factoring:

$x^2 - 2x = 0 \iff x(x - 2) = 0 \iff x = 0$ or $x = 2$.

Warning. Let us have a look on an "alternative solution":

$\cancel{x}\, x\, (x - 1) = \cancel{x}\, x \iff x - 1 = 1 \iff x = 2$.

We have got only one solution, the other solution, $x = 0$, has been **lost**.
The reason for this is an **illegal** cancellation of x .
A cancellation of x is the division by x , which makes sense only if $x \neq 0$.
But $x = 0$ is in fact a solution,
and cancellation of it leads to the loss of this solution.

⚠ Don't cancel anything unknown while solving an equation!

Summary

In this lecture, we have learned

- ☑ what a quadratic polynomial is
- ☑ what the standard form of a quadratic polynomial is $ax^2 + bx + c$
- ☑ why quadratic polynomials are important
- ☑ what a quadratic equation is
- ☑ what it means to solve an equation
- ☑ what the **roots** (or solutions) of a quadratic equation are
- ☑ how to solve a **binomial** quadratic equation

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