Lecture 24

Radicals

Squares and square roots

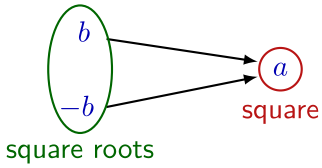
A number and its opposite have the same square:

for example, $3^2 = 9$ and $(-3)^2 = 9$.

Number 9 is called the **square** of 3 (or -3).

Numbers 3 and -3 are called the **square roots** of 9.

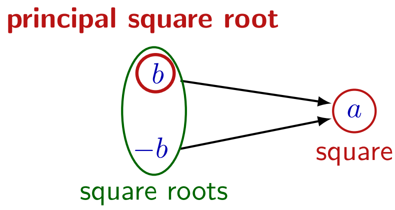
Let a be a non-negative number. A **square root of** a is a number b such that $b^2 = a$.
If a is positive, then there are two numbers, b and $-b$, whose square is a:



If $a = 0$, then there is only one number, 0, whose square is 0: $0 = 0^2$.

Definition of radical

Let a be a non-negative number.
The **principal square root** of a is a non-negative number b such that $b^2 = a$.



Notation for the principal square root: $\sqrt{a} = b$

The symbol $\sqrt{}$ is called a **radical sign**.

The formula $\sqrt{a} = b$ reads "the square root of a is equal to b".

By definition, $\sqrt{a} = b \iff b^2 = a$ for non-negative a and b.

Radicals and perfect squares

Examples. $\sqrt{0} = 0$ since $0^2 = 0$,

$\sqrt{1} = 1$ since $1^2 = 1$,

$\sqrt{4} = 2$ since $2^2 = 4$,

$\sqrt{9} = 3$ since $3^2 = 9$,

$\sqrt{16} = 4$ since $4^2 = 16$.

A number a is called a **perfect square** if $\sqrt{a}$ is an integer.

Here are some perfect squares: $0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100.$

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Precautions

• When we work with **real numbers**, the number under the radical sign should be **non-negative**: $\sqrt{a}$ is defined only for $a \geq 0$.

For example, $\sqrt{-9}$ is **not** defined.

• A square root is always **non-negative**: $\sqrt{a} \geq 0$.

For example, it is **incorrect** to write $\sqrt{9} = -3$, since $\sqrt{9}$, by definition, should be non-negative.

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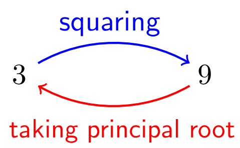
Taking principal square root is opposite to squaring



It means that $\sqrt{3^2} = 3$ and $(\sqrt{9})^2 = 9.$

For any **non-negative** a, $\sqrt{a^2} = a$ and $(\sqrt{a})^2 = a$.

Example. Find the value of the following expressions:

$$\sqrt{5^2}\,,\ \sqrt{(-5)^2}\,,\ \sqrt{-5^2}\,,\ (\sqrt{5})^2\,,\ (-\sqrt{5})^2\,,\ (\sqrt{-5})^2\,.$$

Solution. $\sqrt{5^2} = 5,\quad \sqrt{(-5)^2} = \sqrt{5^2} = 5,\quad \sqrt{-5^2} = \sqrt{-25}$ is undefined

$(\sqrt{5})^2 = 5,\quad (-\sqrt{5})^2 = (\sqrt{5})^2 = 5,\quad (\sqrt{-5})^2$ is undefined

Properties of radicals

Let a, b be non-negative numbers. Then $\sqrt{a}\sqrt{b} = \sqrt{ab}$ and $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

Indeed, $(\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$. Therefore, $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

$\left(\dfrac{\sqrt{a}}{\sqrt{b}}\right)^2 = \dfrac{(\sqrt{a})^2}{(\sqrt{b})^2} = \dfrac{a}{b}$. Therefore, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

Example. Simplify the following expressions: $\sqrt{3}\,\sqrt{12}\,,\quad \sqrt{75}\,,\quad \dfrac{\sqrt{27}}{\sqrt{12}}\,.$

Solution. $\sqrt{3}\,\sqrt{12} = \sqrt{3}\,\sqrt{3\cdot 4} = \sqrt{3}\,\sqrt{3}\,\sqrt{4} = (\sqrt{3})^2\sqrt{2^2} = 3\cdot 2 = 6\,.$

Another way to calculate: $\sqrt{3}\,\sqrt{12} = \sqrt{3\cdot 12} = \sqrt{36} = \sqrt{6^2} = 6\,.$

$\sqrt{75} = \sqrt{3\cdot 25} = \sqrt{3\cdot 5^2} = \sqrt{3}\sqrt{5^2} = \sqrt{3}\cdot 5 = 5\sqrt{3}.$

$\dfrac{\sqrt{27}}{\sqrt{12}} = \dfrac{\sqrt{3\cdot 9}}{\sqrt{3\cdot 4}} = \dfrac{\sqrt{3}\cdot\sqrt{9}}{\sqrt{3}\cdot\sqrt{4}} = \dfrac{\sqrt{3^2}}{\sqrt{2^2}} = \dfrac{3}{2}\,.$

What is $\sqrt{x^2}$?

We know that x^2 is non-negative for any value of x. So $\sqrt{x^2}$ is defined.

Is it true that $\sqrt{x^2} = x$ for **all** x? No!

For **non-negative** x, $\sqrt{x^2} = x$ by definition of the radical.

For **negative** x, $\sqrt{x^2} = -x$, since $-x > 0$ and $(-x)^2 = x^2$.

Therefore, $\sqrt{x^2} = |x|$. Reminder: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Example 1. $\sqrt{(-5)^2} = |-5| = 5$.

Example 2. Simplify the following expressions: $\sqrt{x^4}$, $\sqrt{x^6}$.

Solution. $\sqrt{x^4} = \sqrt{(x^2)^2} = |x^2| = x^2$

$\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3| = |x^2 \cdot x| = |x^2| \cdot |x| = x^2 \cdot |x|$

Why $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$?

It is **not** true that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ for arbitrary x, y.

Indeed, if $x = 9$ and $y = 16$, then

$\sqrt{x+y}\Big|_{x=9,\,y=16} = \sqrt{9+16} = \sqrt{25} = 5$, while

$(\sqrt{x} + \sqrt{y})\Big|_{x=9,\,y=16} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ and $5 \neq 7$.

Are there any x, y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$? Yes!

For example, $x = y = 0$: $\sqrt{0+0} = \sqrt{0} + \sqrt{0}$

or $x = 1$ and $y = 0$: $\sqrt{1+0} = \sqrt{1} + \sqrt{0}$.

Actually, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ only if at least one of x, y is zero.

Simplest radical form

An expression involving radicals can be written in many different forms. For example,

$$\sqrt{\frac{4}{3}} = \frac{\sqrt{4}}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

It is a custom to write radical expressions in a special form, which is called **simplest radical form**.

In simplest radical form, the expression

- doesn't contain perfect square factors:

  $\sqrt{12}$ is not in the simplest form, but $2\sqrt{3}$ is. ($\sqrt{12} = \sqrt{4\cdot 3} = 2\sqrt{3}$)

- doesn't contain fractions under the radical:

  $\sqrt{\frac{3}{4}}$ is not in the simplest form, but $\frac{\sqrt{3}}{2}$ is. $\left(\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}\right)$

- doesn't contain radicals in denominators:

  $\frac{1}{\sqrt{2}}$ is not in the simplest form, but $\frac{\sqrt{2}}{2}$ is. $\left(\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{2}}{2}\right)$

Simplest radical form

Example. Bring the following expressions in simplest radical form:

$$\frac{1}{\sqrt{3}}, \quad \sqrt{\frac{2}{5}}, \quad \frac{1}{3-\sqrt{2}}$$

Solution. $\frac{1}{\sqrt{3}} = \frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{\sqrt{3}}{(\sqrt{3})^2} = \frac{\sqrt{3}}{3}$

$$\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{\sqrt{10}}{(\sqrt{5})^2} = \frac{\sqrt{10}}{5}$$

$$\frac{1}{3-\sqrt{2}} = \frac{1\cdot(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{3+\sqrt{2}}{3^2-(\sqrt{2})^2} = \frac{3+\sqrt{2}}{9-2} = \frac{3+\sqrt{2}}{7}$$

Remember: $(a-b)(a+b) = a^2 - b^2$, so

$$(3-\sqrt{2})(3+\sqrt{2}) = 3^2 - (\sqrt{2})^2$$

Operating with radical expressions

Example 1. Simplify the expression: $\sqrt{6}(\sqrt{18}-\sqrt{24})$

Solution. $\sqrt{6}(\sqrt{18}-\sqrt{24}) = \sqrt{6}\sqrt{18}-\sqrt{6}\sqrt{24} = \sqrt{6\cdot 18}-\sqrt{6\cdot 24} =$

$\sqrt{6\cdot 6\cdot 3}-\sqrt{6\cdot 6\cdot 4} = \sqrt{6^2\cdot 3}-\sqrt{6^2\cdot 2^2} = \sqrt{6^2}\sqrt{3}-\sqrt{6^2}\sqrt{2^2} =$

$6\sqrt{3}-6\cdot 2 = 6\sqrt{3}-12.$

Example 2. Bring the expression in simplest radical form: $\dfrac{\sqrt{6}-3}{\sqrt{3}-\sqrt{2}}$.

Solution.

$$\frac{\sqrt{6}-3}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3\cdot 2}-(\sqrt{3})^2}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}-(\sqrt{3})^2}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}(\sqrt{2}-\sqrt{3})}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{\sqrt{3}(-1)(\sqrt{3}-\sqrt{2})}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}(-1)(\cancel{\sqrt{3}-\sqrt{2}})}{\cancel{\sqrt{3}-\sqrt{2}}} = -\sqrt{3}.$$

Summary

In this lecture, we have learned

- ☑ what the **square roots** of a non-negative number are
- ☑ what the **principal square root** is
- ☑ what the perfect squares are
- ☑ the defining identities for radical: $\sqrt{a^2}=a$ and $(\sqrt{a})^2=a$
- ☑ the properties of radicals: $\sqrt{a}\sqrt{b}=\sqrt{ab}$, $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$
- ☑ $\sqrt{x^2}=|x|$ that for all x
- ☑ $\sqrt{x+y}\neq\sqrt{x}+\sqrt{y}$ for arbitrary x, y
- ☑ what the simplest radical form is
- ☑ how to operate with radical expressions