Lecture 22

Linear Systems. Part 2

Preface

In Lecture 21, we learned

- what a linear system is
- what its solution is
- how many solutions a system may have
- how to solve a system by **elementary transformations:**
  adding/subtracting equations and
  multiplying an equation by a non-zero number.

We continue our journey through the theory shifting the attention to examples.

We will solve one by one specific systems,
gradually learning new practical tricks and fragments of theory.

Substitution

Example 1. Solve the system $\begin{cases} x - 3y = 1 \\ y = 2. \end{cases}$

It's a nice system: the second equation says the unknown y is actually **known**!
Solution: Plug $y = 2$ into the first equation:

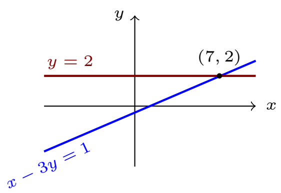
$$\begin{cases} x - 3y = 1 \\ y = 2 \end{cases} \iff \begin{cases} x - 3(2) = 1 \\ y = 2 \end{cases} \iff \begin{cases} x = 1 + 6 \\ y = 2 \end{cases} \iff \begin{cases} x = 7 \\ y = 2 \end{cases}$$

This method is called substitution.

This system could be solved also by elementary transformations:

$$\begin{cases} x - 3y = 1 \\ y = 2 \end{cases} \iff \begin{cases} x - 3y = 1 \\ 3y = 6 \end{cases} \iff \begin{cases} x = 7 \\ 3y = 6 \end{cases} \iff \begin{cases} x = 7 \\ y = 2 \end{cases}$$

Geometric interpretation:



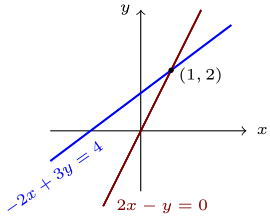
Elimination by addition

Example 2. Solve the system $\begin{cases} -2x + 3y = 4 \\ 2x - \ \ y = 0 \end{cases}$

Solution.
The coefficients for x are -2 and 2, so adding the equations will **eliminate** x:

$$\begin{cases} -2x + 3y = 4 \\ 2x - \ \ y = 0 \end{cases} \iff \begin{cases} 2y = 4 \\ 2x - \ \ y = 0 \end{cases} \iff \begin{cases} y = 2 \\ 2x - y = 0 \end{cases} \iff \begin{cases} y = 2 \\ 2x \quad = 2 \end{cases}$$

$$\iff \begin{cases} y = 2 \\ x = 1 \end{cases} \iff (x, y) = (1, 2)$$



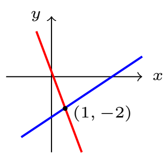
All methods together

Example 3. Solve the system $\begin{cases} -2x + 3y = -8 \\ 5x + 2y = 1 \end{cases}$

Solution.
Let us **eliminate** one of the unknowns, say x:

$$\begin{cases} -2x + 3y = -8 \\ 5x + 2y = 1 \end{cases} \begin{array}{c} \xrightarrow{\text{multiply by } 5} \\ \xrightarrow{\text{multiply by } 2} \end{array} \begin{cases} -10x + 15y = -40 \\ 10x + \ \ 4y = 2 \end{cases} \iff \begin{cases} 19y = -38 \\ 10x + 4y = 2 \end{cases}$$

$$\iff \begin{cases} y = -2 \\ 5x + 2y = 1 \end{cases} \iff \begin{cases} y = -2 \\ 5x + 2(-2) = 1 \end{cases} \iff (x, y) = (1, -2)$$



How to check a solution?

It is easy to **check** if a solution of a linear system is correct.

Let us check if $(x, y) = (1, -2)$ is indeed a correct solution of the system

$$\begin{cases} -2x + 3y = -8 \\ 5x + 2y = 1 \end{cases}$$

Plug in $x = 1$, $y = -2$ into the system:

$$\begin{cases} -2(1) + 3(-2) \overset{?}{=} -8 \\ 5(1) + 2(-2) \overset{?}{=} 1 \end{cases} \iff \begin{cases} -8 \overset{\checkmark}{=} -8 \\ 1 \overset{\checkmark}{=} 1 \end{cases}$$

Systems with no solutions

Solve the system $\begin{cases} x + 2y = -1 \\ -2x - 4y = 3. \end{cases}$

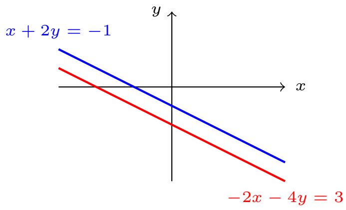
Solution.

$$\begin{cases} x + 2y = -1 \\ -2x - 4y = 3 \end{cases} \iff \begin{cases} 2x + 4y = -2 \\ -2x - 4y = 3 \end{cases} \iff \begin{cases} 2x + 4y = -2 \\ 0 = 1 \end{cases}$$

The statement $0 = 1$ is **false**.

It is false **no matter** what values x and y take.

A system, which includes an equation $0 = 1$, has **no** solution.



Systems with infinitely many solutions

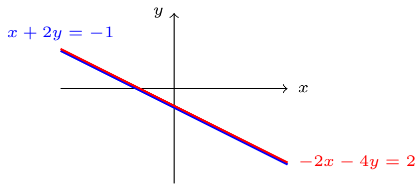
Solve the system $\begin{cases} x + 2y = -1 \\ -2x - 4y = 2. \end{cases}$

Solution.

$$\begin{cases} x + 2y = -1 \\ -2x - 4y = 2 \end{cases} \iff \begin{cases} 2x + 4y = -2 \\ -2x - 4y = 2 \end{cases} \iff \begin{cases} 2x + 4y = -2 \\ 0 = 0 \end{cases}$$

The statement $0 = 0$ is **true**. It is true, **no matter** what values x and y take. Removing the equation $0 = 0$ from a system does not change the set of solutions. Our system is equivalent to a single equation:

$$2x + 4y = -2 \iff x + 2y = -1 \iff x = -1 - 2y$$



Answer: $(x, y) = (-1 - 2y,\, y)$, where y is an arbitrary number.

Summary

In this lecture, we have learned

- ☑ how to solve a system by a **substitution**
- ☑ how to **eliminate** an unknown
- ☑ how to **check** a solution
- ☑ how to handle systems with **no solutions**
- ☑ how to handle systems with **infinitely many solutions**