Lecture 20

Lines on a Plane. Part 2

Linear equation $y = mx + b$

Consider a general linear equation $Ax + By = C$ whose graph is a line.

If $B \neq 0$, then the equation can be rewritten as follows:

$$Ax + By = C \iff By = -Ax + C \iff y = -\frac{A}{B}x + \frac{C}{B} \iff y = mx + b,$$

where $m = -\frac{A}{B}$ and $b = \frac{C}{B}$.

If $B = 0$, then $Ax + By = C \iff Ax = C \iff x = \frac{C}{A}$,
and the graph is a **vertical** line.

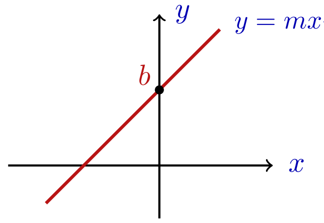
Any non-vertical line can be described by the equation $y = mx + b$.

The y-intercept

Consider a linear equation $y = mx + b$. What do the coefficients m and b represent?

The coefficient b represents the y-**intercept** of the line $y = mx+b$.

Indeed, if $x = 0$ then $y = m \cdot 0 + b \iff y = b$ and $(0,b)$ is the y-**intercept**.



The coefficient b in the equation $y = mx+b$ shows where the line meets the y-**axis**.

The coefficient b is called the y-**intercept**.

Slope-intercept equation of a line

The coefficient $\boldsymbol{m}$ in the equation $y = \boldsymbol{m}x + b$ is called the **slope** of the line.

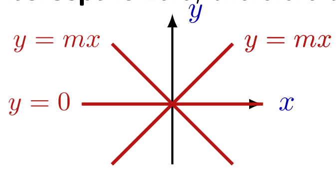
y-intercept
↓
$$y = \boldsymbol{m}x + b$$
↑
slope

The equation $\boxed{y = mx + b}$ is called the **slope-intercept** equation of a line.

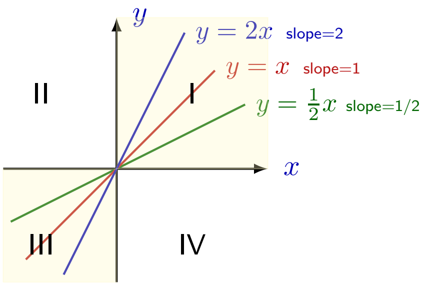
What does the slope of the line represent?

Slope measures the inclination of a line

Let us study a line $y = mx$. The y-**intercept** is zero, therefore the line passes through the origin.



Here are several lines with **positive** slopes:



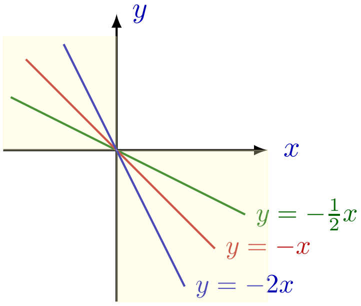
The larger the slope, the steeper the line.

A line with positive slope **rises** as we move from left to right.

Lines $y = mx$ with **positive** m are located in the **first** and **third** quadrants of the plane.

Negative slope. Zero slope

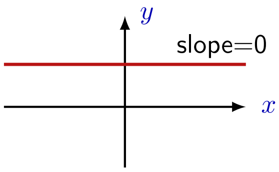
Here are several lines with **negative** slopes:



A line with negative slope **falls** as we move from left to right.

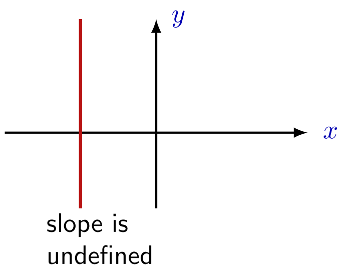
A line $y = mx$ with **negative** m is located in the **second** and **fourth** quadrants.

If the slope $m = 0$, then
$y = mx + b \iff y = 0 \cdot x + b \iff y = b,$
and the line is **horizontal**.



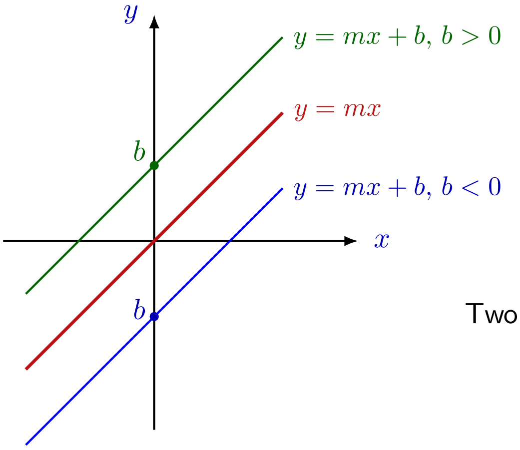
Slope of vertical line

The slope of a vertical line is **undefined**.



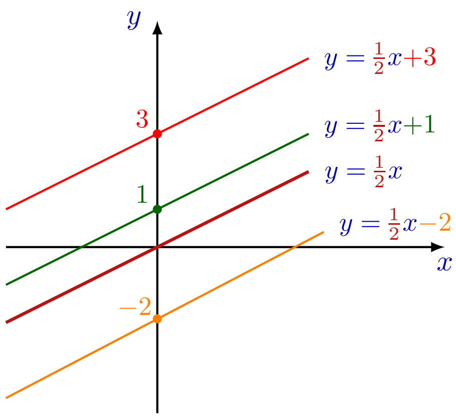
Parallel lines have the same slope

A line $y = mx + b$ is obtained from the line $y = mx$ by a **vertical shift** along the y-axis.



Two non-vertical lines are **parallel**
if and only if
they have **the same** slope.

Example of parallel lines



Parallel or not?

Example 1. Are the lines $3x - 2y = 1$ and $-6x + 4y = 5$ parallel?

Solution. To answer the question, we have to determine the **slopes** of the lines. For this, we rewrite the equations in the **slope-intercept** form $y = mx + b$.

$$3x - 2y = 1 \iff 2y = 3x - 1 \iff y = \frac{3}{2}x - \frac{1}{2}$$

$$-6x + 4y = 5 \iff 4y = 6x + 5 \iff y = \frac{6}{4}x + \frac{5}{4} \iff y = \frac{3}{2}x + \frac{5}{4}.$$

Since the lines have the same slope of $\frac{3}{2}$, they are **parallel**.

Example 2. Are the lines $y = 2$ and $y = 2x$ parallel?

Solution. The slope of the line $y = 2$ is 0, since $y = 2 \iff y = 0 \cdot x + 2$.
The slope of line $y = 2x$ is 2. Since the lines have different slopes, they are **not** parallel.
Remark. $y = 2$ is a **horizontal** line, while $y = 2x$ is not. So the lines are not parallel.

Slope of a line through two given points

Theorem. A line passing though two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ has the slope $\frac{y_2 - y_1}{x_2 - x_1}$.

Proof. Let $y = mx + b$ be an equation of the line. We have to prove that the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.
Since the points (x_1, y_1) and (x_2, y_2) are on the line $y = mx + b$,
their coordinates satisfy the equation $y = mx + b$:

$$y_1 = mx_1 + b \quad \text{and} \quad y_2 = mx_2 + b.$$

Subtracting the first equality from the second one, we get

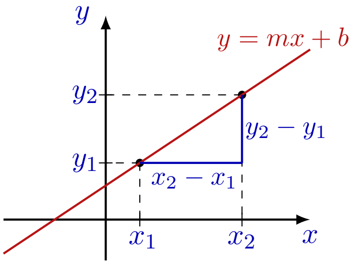
$$y_2 - y_1 = (mx_2 + b) - (mx_1 + b) \iff y_2 - y_1 = m(x_2 - x_1) \iff m = \frac{y_2 - y_1}{x_2 - x_1}.$$ as required.

Notice that $x_2 - x_1 \neq 0$ since $x_1 \neq x_2$.

Slope as a ratio

Let us give a **geometric** interpretation of this result:

A line $y = mx + b$ passing through the points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ has the slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



When we move along the line from a point (x_1, y_1) to another point (x_2, y_2), the difference $x_2 - x_1$ shows the change in x-coordinate, and the difference $y_2 - y_1$ shows the change in y-coordinate.

The slope is the ratio of the change: $\text{slope} = \dfrac{\text{change in } y}{\text{change in } x}$

Examples

Example 1. Find the equation of the line passing through the points $(1, -1)$ and $(-3, 7)$.
Solution. Let $y = mx + b$ be the equation of the line.
We have to determine the coefficients m and b.

The slope m of the line passing through the points $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-3, 7)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-1)}{-3 - 1} = \frac{8}{-4} = -2.$$

Our line has the equation $y = -2x + b$.
To determine b, we plug in any of two given points into this equation.
Plugging in $(x_1, y_1) = (1, -1)$, we get

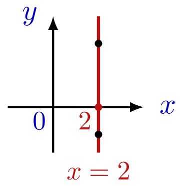
$$\underbrace{-1}_{y_1} = \underbrace{-2}_{m} \cdot \underbrace{1}_{x_1} + b \iff -1 = -2 + b \iff b = 1.$$

Therefore, the line has equation $\boxed{y = -2x + 1}$

Examples

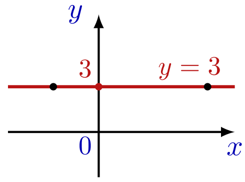
Example 2. Find the equation of the line passing through the points $(2,-1)$ and $(2,3)$.

Solution. The given points have the **same** x-coordinate.
Therefore, they belong to a **vertical** line.
The equation of the line is $x=2$.



Example 3. Find the equation of the line passing through the points $(-1,3)$ and $(4,3)$.

Solution. The given points have the **same** y-coordinate.
Therefore, they belong to a **horizontal** line.
The equation of the line is $y=3$.



Point-slope equation

Theorem. A line that has a slope of m and passes through the point (x_1,y_1) has the equation $y-y_1=m(x-x_1)$

Proof. Let us show that the equation above describes a line that has a slope of m and passes through (x_1,y_1).
Rewrite the equation in a slope-intercept form:

$$y-y_1=m(x-x_1) \iff y=\underbrace{m}_{slope}x+(-mx_1+y_1).$$

The coefficient in front of x is the slope m.
Moreover, the point (x_1,y_1) satisfies the equation $y-y_1=m(x-x_1)$:
$y_1-y_1=m(x_1-x_1) \iff 0=0$, so it belongs to the line.

Example. Find a slope-intercept equation of a line that has a slope of 3 and passes through the point $(-1,2)$.

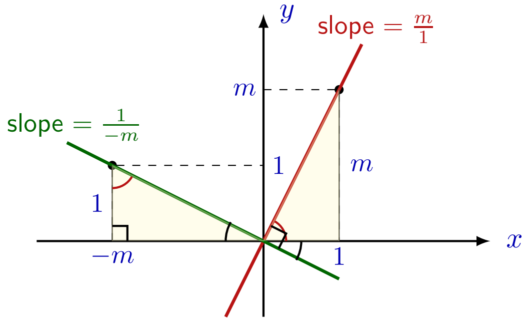
Solution. Using the point-slope equation $y-y_1=m(x-x_1)$, we get

$$y-2=3(x-(-1)) \iff y-2=3(x+1) \iff \boxed{y=3x+5}$$

Perpendicular lines

Theorem. Two non-vertical lines are perpendicular if the product of their slopes is -1.

Proof.



The triangles are congruent.
The lines are perpendicular.

Example. Prove that the lines $x - 2y = 1$ and $6x + 3y = 2$ are perpendicular.

Solution. $x - 2y = 1 \iff 2y = x - 1 \iff y = \frac{1}{2}x - 1/2$

$6x + 3y = 2 \iff 3y = -6x + 2 \iff y = -2x + 2/3$

The slopes $1/2$ and -2 are negative reciprocals of each other.

Therefore, the lines are perpendicular.

Summary

In this lecture, we have learned

- ☑ the **slope-intercept equation** of a line $y = mx + b$
- ☑ what the **slope** of a line represents
- ☑ that **parallel lines** have the same slope
- ☑ how to find equation of a line passing through two points
- ☑ what the **point-slope equation** of a line is $y - y_1 = m(x - x_1)$
- ☑ that **perpendicular lines** have negative reciprocals slopes