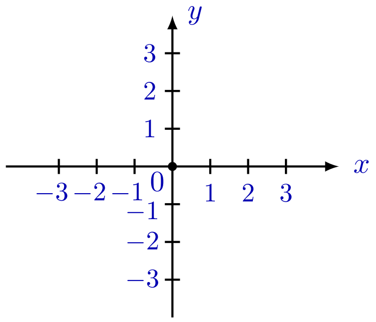
Lecture 19

Lines on a Plane. Part 1

Cartesian coordinate system on a plane

Cartesian (or **rectangular**) coordinate system is defined by
• a point, called the **origin**,
• two perpendicular **number lines** drawn through the origin.

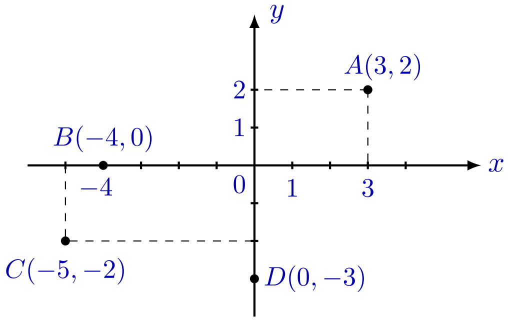


Usually, one line is drawn horizontally, and the other one vertically.

The horizontal line is called $\boldsymbol{x}$-**axis**, the vertical line is called the $\boldsymbol{y}$-**axis**.

Points and their coordinates

Given the coordinate system, each point on the plane gets its **coordinates** –
two numbers which determine the location of the point on the plane.

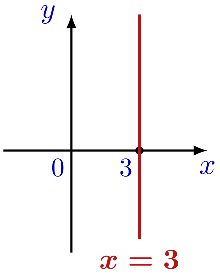


The first number is called $\boldsymbol{x}$-**coordinate**, the second number is called the $\boldsymbol{y}$-**coordinate**.
For example, the coordinates of point A are $(3, 2)$,
where 3 is the x-coordinate and 2 is the y-coordinate.

Vertical lines

Example. Describe geometrically the set of all points on the coordinate plane whose x-coordinate is 3.

Solution. These are the points with coordinates $(3, y)$, where y is an arbitrary number.



All such points form a **vertical** line passing through the point 3 on the x-axis.

This vertical line is the **graph** of the equation $\boldsymbol{x = 3}$.

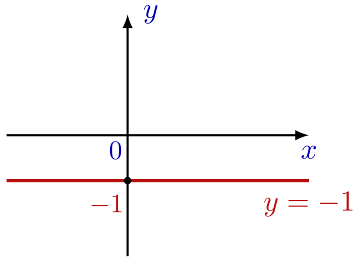
The graph of the equation $x = a$, where a is a number, is the **vertical** line passing through the point a on the x-axis.

The y-axis, which is a vertical line, has the equation $x = 0$.

Horizontal lines

Example. Draw the graph of the equation $y = -1$.

Solution. The graph of $y = -1$ is the set of all points on the plane whose coordinates are $(x, -1)$, where x is an arbitrary number.
It is a **horizontal** line passing through the point -1 on the y-axis.



The graph of the equation $y = b$, where b is a number, is the **horizontal** line passing through the point b on the y-axis.

The x-axis, which is a horizontal line, has the equation $y = 0$.

General linear equation in two variables

The equation $Ax + By = C$, where A, B, C are given numbers and x, y are variables, is called a **linear equation** in two variables.

The numbers A, B, C are called the **coefficients**.

Examples of linear equations in two variables:

$-2x + y = 4 \quad (A = -2,\ B = 1,\ C = 4)$,

$x = 1 \iff x + 0 \cdot y = 1 \quad (A = 1,\ B = 0,\ C = 1)$,

$y = 0 \iff 0 \cdot x + y = 0 \quad (A = 0,\ B = 1,\ C = 0)$,

$0 = 3 \iff 0 \cdot x + 0 \cdot y = 3 \quad (A = 0,\ B = 0,\ C = 3)$,

$0 = 0 \iff 0 \cdot x + 0 \cdot y = 0 \quad (A = 0,\ B = 0,\ C = 0)$.

The **graph** of an equation is the set of **all** points on the plane whose coordinates satisfy the equation.

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The graph of a linear equation in two variables

What is the **graph** of the equation $Ax + By = C$? It depends on the coefficients A, B, C.

• If all the coefficients are zeros, that is $A = B = C = 0$, then the equation is

$$0 \cdot x + 0 \cdot y = 0 \iff 0 = 0,$$

and it is satisfied by **any** pair of numbers (x, y). Therefore, its graph is the **entire plane**.

• If $A = B = 0$ and $C \neq 0$ then the equation is

$$0 \cdot x + 0 \cdot y = C \iff 0 = C\,,$$

and there are **no** (x, y) satisfying it. Its graph is the **empty set**.

• If $A,\ B$ are **not** both zero, that is either $A \neq 0$ or $B \neq 0$, then the graph is a **straight line**.

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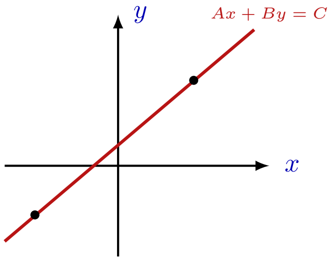
Line as the graph of a linear equation

If A, B are **not** both zero, then there are **infinitely many** points (x, y) satisfying the equation $Ax + By = C$.

They are located on a **straight line**. This line is the **graph** of the equation $Ax + By = C$.



A line is determined by any two of its points. Therefore, to draw the line, it is enough to specify the location of **two** points on it.

How to draw a line by its equation

Example. Draw the line $3x - 4y = 12$ on the coordinate plane.

Solution. Let us pick up two points on the line. A point on the line is defined by a pair of numbers (x, y), satisfying the equation $3x - 4y = 12$.

For simplicity, let us choose $x = 0$. Then

$$3 \cdot 0 - 4y = 12 \iff -4y = 12 \iff y = -3 .$$

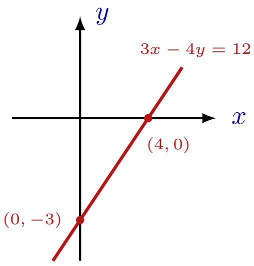
Therefore, $(0, -3)$ is a point on the line.

Now put $y = 0$. Then

$$3x - 4 \cdot 0 = 12 \iff 3x = 12 \iff x = 4 .$$

Therefore, $(4, 0)$ is a point on the line.

Draw a line through $(0, -3)$ and $(4, 0)$:



A line through two points

Remark. When we search for two points belonging to the line $3x - 4y = 12$,
it is convenient to put the coordinates in the table:

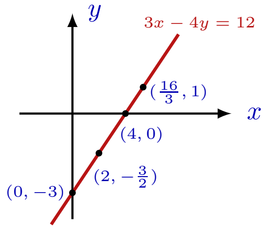
x	y
0	-3
4	0

$$x = 0 \implies y = -3$$
$$y = 0 \implies x = 4$$

One may choose any two other points on the line, for example,

x	y
2	$-\frac{3}{2}$
$\frac{16}{3}$	1

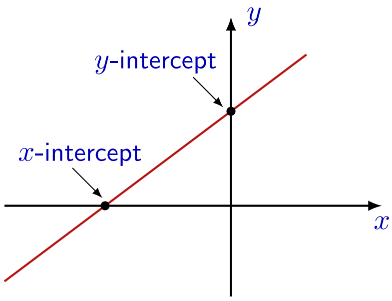
$$x = 2 \implies 3 \cdot 2 - 4y = 12 \implies 6 - 4y = 12 \implies y = -\frac{3}{2}$$
$$y = 1 \implies 3x - 4 \cdot 1 = 12 \implies 3x = 16 \implies x = \frac{16}{3}$$



Intercepts

The point where the line intersects the x-axis is called the x-**intercept**.
The x-intercept has coordinates $(x, 0)$, its y-coordinate equals 0.

The point where the line intersects the y-axis is called the y-**intercept**.
The y-intercept has coordinates $(0, y)$, its x-coordinate equals 0.

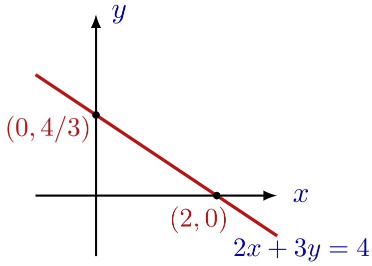


How to find intercepts

Example. Determine the intercepts of the line $2x + 3y = 4$.
Draw the line on the coordinate system.

Solution.
The x-intercept is the point where $y = 0$. Plug in $y = 0$ into the equation:
$2x + 3 \cdot 0 = 4 \iff 2x = 4 \iff x = 2$. So the x-intercept is $(2, 0)$.

The y-intercept is the point where $x = 0$. Plug in $x = 0$ into the equation:
$2 \cdot 0 + 3y = 4 \iff 3y = 4 \iff y = 4/3$. So the y-intercept is $(0, 4/3)$.



Two-intercept form of a linear equation

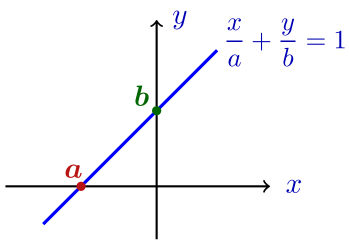
The equation $\boxed{\dfrac{x}{a} + \dfrac{y}{b} = 1}$ where x, y are variables and a, b are non-zero numbers,

is called the **two-intercept** equation of a line.

The coefficients $\boldsymbol{a}$ and $\boldsymbol{b}$ represent the x- and y-intercepts respectively.

Indeed, $(\boldsymbol{a}, \mathbf{0})$ and $(\mathbf{0}, \boldsymbol{b})$ satisfy the equation:

$$\frac{\boldsymbol{a}}{a} + \frac{\mathbf{0}}{b} = 1 \quad \text{and} \quad \frac{\mathbf{0}}{a} + \frac{\boldsymbol{b}}{b} = 1\,.$$



Quick drawing

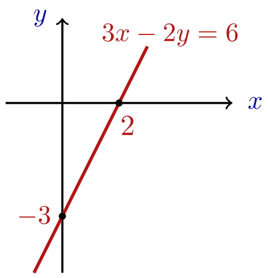
The two intercept form of the equation helps to draw a line in no time.

Example. Draw the line $3x - 2y = 6$.

Solution. Rewrite the equation in the **two-intercept** form:

$$3x - 2y = 6 \iff \frac{3x}{6} - \frac{2y}{6} = 1 \iff \frac{x}{2} + \frac{y}{-3} = 1.$$

The x-intercept is $(2,0)$, the y-intercept is $(0,-3)$.



Summary

In this lecture, we have learned

- ☑ what a **Cartesian** coordinate system is
- ☑ what the equation of a **vertical** line is ($x = a$)
- ☑ what the equation of a **horizontal** line is ($y = b$)
- ☑ what the general **linear equation in two variables** is ($Ax + By = C$)
- ☑ what the **graph** of a linear equation is
- ☑ how to draw a line by its equation
- ☑ what the **intercepts** are
- ☑ what the **two-intercept** equation is $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$