Lecture 17

# Linear Inequalities

## What a linear inequality is

There are four **inequality signs**: $<$, $\leq$, $>$, $\geq$.

| | |
|---|---|
| $a < b$ | $a$ is **less** than $b$ |
| $a \leq b$ | $a$ is **less** than **or equal** to $b$ |
| $a > b$ | $a$ is **greater** than $b$ |
| $a \geq b$ | $a$ is **greater** than **or equal** to $b$ |

A **linear inequality** consists of two linear expressions connected by one of the inequality signs.

For example, $3(x-1) \leq 4+5x$ is a linear inequality in one variable.

Evaluation of both sides of an inequality at a **number**
gives rise to a numerical inequality, which may be either true or false.

For example, at $x = 0$ the inequality above holds true:

$$3(0-1) \leq 4 + 5 \cdot 0 \iff -3 \leq 4 \quad \checkmark$$



## Solution

**To solve** an inequality means to find **all** values of the variable, for which the inequality holds true.

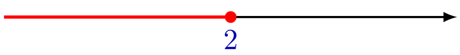
These values form a **solution set**.

A linear **inequality** is very similar to a linear **equation**.

As we remember, the solution set of a linear **equation**
• either consists of a single number (when the equation has one solution),
• or is **empty** (when the equation has no solutions),
• or is the entire number line (when the equation has infinitely many solutions).

The solution set of a linear inequality is quite different.

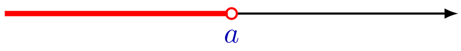
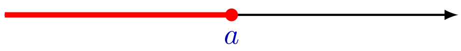
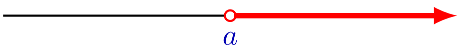
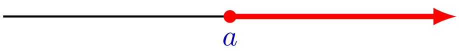
Consider a simple inequality $x \leq 2$. Its solution set consists of all numbers $\leq 2$
and is denoted by $\{x \mid x \leq 2\}$. One can **graph** the solutions on the number line:




The solution set is an **interval**. It is denoted by $(-\infty, 2]$.

## Intervals

Let us review intervals that we may encounter solving linear inequalities.

| inequality | solution | graph | interval |
|---|---|---|---|
| $x < a$ | $\{x \mid x < a\}$ |   | $(-\infty, a)$ |
| $x \leq a$ | $\{x \mid x \leq a\}$ |   | $(-\infty, a]$ |
| $x > a$ | $\{x \mid x > a\}$ |   | $(a, \infty)$ |
| $x \geq a$ | $\{x \mid x \geq a\}$ |   | $[a, \infty)$ |



## Equivalent inequalities

Two inequalities are called **equivalent** if they have the same solution sets.

It means that each solution of the first inequality is a solution of the second one, and vice versa: each solution of the second inequality is a solution of the first one.

If two inequalities are equivalent, we write the equivalence sign " $\iff$ " between them, like this

$$x + 1 > 3 \iff x > 2\,.$$

How to **transform** an inequality into an equivalent inequality?

To this end, we will use three **elementary** transformations.

## Add the same to both sides

> Any inequality is equivalent to the inequality obtained from it by **adding** the same expression to **both** sides.

**Example 1.** Consider the inequality $x - 1 > 2$. If we add $1$ to both sides, then we get an equavalent inequality:

$$x - 1 > 2 \iff x - 1 + 1 > 2 + 1 \iff \boxed{x > 3}$$

**Example 2.** $5 - x \le 0 \iff 5 - x + x \le 0 + x \iff 5 \le x \iff \boxed{x \ge 5}$

**Example 3.**

$$5 - x < 2 \iff 5 - x + (x - 2) < 2 + (x - 2) \iff$$
$$5 - x + x - 2 < 2 + x - 2 \iff 3 < x \iff \boxed{x > 3}$$

Similarly, **subtracting** the same expression from both sides of an inequality gives rise to an equivalent inequality:

$$x + 2 \ge 6 \iff x + 2 - 2 \ge 6 - 2 \iff \boxed{x \ge 4}$$



## Fast track

There is a trick that may help you to operate **more efficiently** with inequalities.

The subtraction of $x$ from both sides of the inequality $2x - 1 \le 5 + x$, namely

$$2x - 1 \le 5 + x \iff 2x - 1 - x \le 5 + x - x \iff x - 1 \le 5$$

is equivalent to relocation $x$ from the right hand side (RHS) of the inequality to the left hand side (LHS) with the **opposite** sign:

$$2x - 1 \le 5 + x \iff 2x - x - 1 \le 5 \iff x - 1 \le 5$$

Look how **fast** we can solve the inequality:

$$2x - 1 \le 5 + x \iff x - 1 \le 5 \iff x \le 6\,.$$

## Multiply both sides by the same positive number

Any inequality is equivalent to the inequality obtained from it by **multiplying** both sides by the same **positive** number.

**Example 1.** $\frac{x}{2} > 3 \iff \frac{x}{2} \cdot 2 > 3 \cdot 2 \iff \boxed{x > 6}$

**Example 2.** $3x \leq 5 \iff 3x \cdot \frac{1}{3} \leq 5 \cdot \frac{1}{3} \iff \boxed{x \leq \frac{5}{3}}$

Similarly, **dividing** both sides of an inequality by the same **positive** number gives rise to an equivalent inequality:

$$2x \geq 8 \iff \frac{2x}{2} \geq \frac{8}{2} \iff \boxed{x \geq 4}$$



## Multiply by negative number and reverse the sign

What happens if we multiply an inequality by a **negative** number?

Consider the inequality $x > 2$. Move $x$ to RHS, and move $2$ to LHS (don't forget to change the signs):

$$x > 2 \iff -2 > -x\,.$$

This inequality says that $-2$ is greater than $-x$. This is the same as $-x$ is less than $-2$:

$$-2 > -x \iff -x < -2\,.$$

Therefore, $x > 2 \iff -x < -2\,.$

In general, if we multiply both sides of an inequality by a **negative** number, we have to **reverse the sign** of the inequality.

**Example 1.** $-\frac{x}{3} < 2 \iff (-3) \cdot \left(-\frac{x}{3}\right) > (-3) \cdot 2 \iff x > -6.$

The same rule is valid if we **divide** an inequality by a negative number.

**Example 2.** $-2x \leq 6 \iff \frac{-2x}{-2} \geq \frac{6}{-2} \iff x \geq -3\,.$

## Elementary transformations

**Elementary transformations** of an inequality are

- **adding** the same expression to both sides of an inequality,
- **multiplying** both sides by the the same **positive** number, and
- **multiplying** both sides by the the same **negative** number and **reversing** the sign of the inequality.

See how a sequence of **elementary transformations** brings an inequality to a simple equivalent inequality.



## Examples

**Example 1.** Solve the inequality $7x - 5 \leq 2x + 1$. Give the answer in interval notation. Show the solution on the number line.

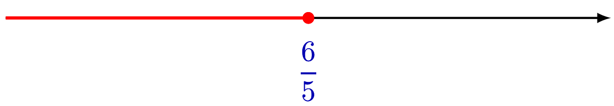
**Solution.** Move $2x$ to the LHS: $7x - 2x - 5 \leq 1$

Simplify: $5x - 5 \leq 1$

Move $-5$ to the RHS: $5x \leq 1 + 5$

Simplify: $5x \leq 6$

Divide by $5$: $\boxed{x \leq \frac{6}{5}}$




**Answer.** $\left(-\infty, \frac{6}{5}\right]$

## Examples

**Example 2.** Solve the inequality $-\frac{x}{2}+3<x+4$. Give the answer in interval notation. Show the solution on the number line.

**Solution.**

Move $3$ to the RHS: $-\frac{x}{2}<x+4-3$

Simplify: $-\frac{x}{2}<x+1$

Multiply by $(-2)$: $(-2)\left(-\frac{x}{2}\right)>(-2)(x+1)$

Simplify: $x>-2x-2$

Move $-2x$ to the LHS: $x+2x>-2$

Simplify: $3x>-2$

Divide by $3$: $x>-\frac{2}{3}$

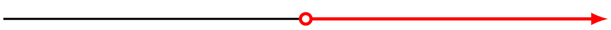


## Writing down the answer

The answer can be written as an inequality $x>-\frac{2}{3}$,

or as a set $\left\{x \mid x>-\frac{2}{3}\right\}$,

or as an interval $\left(-\frac{2}{3},\infty\right)$ on a number line:



$-\frac{2}{3}$

## Systems of linear inequalities

Two inequalities with the same single variable may form a **system**.

To **solve** a system means

to find all the values of the variable that satisfy **both** inequalities.

**Example.** Solve the system $\begin{cases} 3x-2 \le 2x-1 \\ -2x+3<4. \end{cases}$

Write the answer in interval notation. Show the solution on the number line.

**Solution.**

$$\begin{cases} 3x-2 \le 2x-1 \\ -2x+3<4 \end{cases} \iff \begin{cases} 3x-2x \le -1+2 \\ -2x<1 \end{cases} \iff \begin{cases} x \le 1 \\ x > -\dfrac{1}{2} \end{cases} \iff -\frac{1}{2} < x \le 1$$


**Answer:** $\left(-\frac{1}{2}, 1\right]$



## Solution of a system

Geometrically, the solution of a system of two linear inequalities in one variable

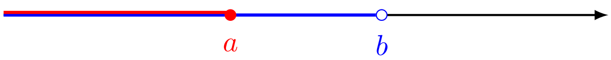
is the **intersection** of two intervals.

The intersection consists of all points belonging to **both** intervals.

As the intersection, we may get a **finite interval**, for example, $a \le x < b$:

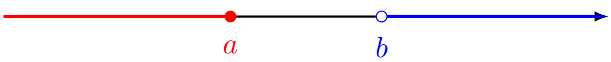

$[a, b)$

an **infinite interval**, for example $x \le a$:




$(-\infty, a]$

or the **empty set** (when the system has no solutions):

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**Summary**

In this lecture, we have learned

- ☑ what a **linear inequality** is
- ☑ what the **solution** of an inequality is
- ☑ which **intervals** on a real line may appear as solutions of inequalities
- ☑ which inequalities are called **equivalent**
- ☑ what **elementary transformations** of inequalities are
  - adding the same expression to both sides
  - multiplying both sides by the same **positive** number
  - multiplying both sides by the same **negative** number and **reversing the sign** of the inequality
- ☑ how to solve inequalities **efficiently**
- ☑ how to **write down** the solution of an inequality
- ☑ how to show the solution on a **number line**
- ☑ how to solve a **system** of inequalities