Lecture 11

Rational Expressions

What a rational expressions is

A **rational expression** $\frac{p}{q}$ is a quotient of two polynomials p and q, where q is **non**-zero polynomial.

For example, $\frac{x+1}{x^2}$, $\frac{3x^3-x^2+x}{x^2+3x-2}$, $\frac{x}{1}$, $\frac{xy+2}{x^2+y^2}$ are rational expressions.

Any polynomial $p(x)$ is a rational expression whose denominator is 1:

$$p(x) = \frac{p(x)}{1}.$$

In this lecture, we will learn how to:

- **evaluate** a rational expression at a number
- **substitute** an expression into a rational expression
- **simplify** rational expressions

Evaluation

Example. Find the value of the expression $\frac{-x^2+4}{x-3}$ for $x=1$, $x=-1$, $x=3$.

Solution. We have to substitute $x = 1, \ -1, \ 3$ into the expression.

$$\left.\frac{-x^2+4}{x-3}\right|_{x=1} = \frac{-(1)^2+4}{(1)-3} = \frac{-1+4}{1-3} = \frac{3}{-2} = -\frac{3}{2}.$$

$$\left.\frac{-x^2+4}{x-3}\right|_{x=-1} = \frac{-(-1)^2+4}{(-1)-3} = \frac{-1+4}{-1-3} = \frac{3}{-4} = -\frac{3}{4}.$$

$$\left.\frac{-x^2+4}{x-3}\right|_{x=3} = \frac{-(3)^2+4}{(3)-3} = \frac{-9+4}{0}$$ Oops! Division by 0 is prohibited!

Therefore, the expression $\frac{-x^2+4}{x-3}$ is **not** defined for $x=3$.

Substitution

Example 1. Find the value of the expression $\frac{x-1}{x^2+2x}$ for $x = a - 1$.

Solution. We have to substitute $a-1$ for x into the expression $\frac{x-1}{x^2+2x}$.
The result should be a new expression involving a, not x.

$$\left.\frac{x-1}{x^2+2x}\right|_{x=a-1} = \frac{(a-1)-1}{(a-1)^2+2(a-1)} = \frac{a-1-1}{a^2-2a+1+2a-2} = \frac{a-2}{a^2-1}.$$

Short multiplication: $(a-1)^2 = a^2 - 2a + 1$

Example 2. Find the value of the expression $\frac{1}{xy}$ for $x = a^2$ and $y = a^{-3}$.

Solution. $\left.\frac{1}{xy}\right|_{x=a^2,\, y=a^{-3}} = \frac{1}{a^2 a^{-3}} = \frac{1}{a^{2-3}} = \frac{1}{a^{-1}} = a.$

Cancellation

Cancellation rule says that

one can cancel out a **common factor** both in numerator and denominator:

$$\frac{ac}{bc} = \frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}.$$

Examples. $\frac{(x+1)(x-1)}{x+1} = \frac{\cancel{(x+1)}\cdot(x-1)}{\cancel{(x+1)}\cdot 1} = \frac{x-1}{1} = x-1.$

$$\frac{x^2\cdot(x+1)^3}{x^5\cdot(x+1)^2} = \frac{x^2\cdot(x+1)^2\cdot(x+1)}{x^2\cdot x^3\cdot(x+1)^2} = \frac{x+1}{x^3}.$$

Warning: It's **incorrect** to cancel out a common summand:

$$\frac{a+c}{b+c} \neq \frac{a}{b}.$$

For example, $\frac{4}{5} = \frac{1+3}{2+3} \neq \frac{1}{2}.$

Cancellation simplifies

Factoring followed by cancellation is used to **simplify** rational expressions.

Example. Simplify the expression $\frac{x^2 - x}{x^2 - 1}$.

Solution. Both numerator and denominator may by factored:

In numerator $x^2 - x$, we factor out x:

$$x^2 - x = x(x-1).$$

To factor denominator, we use the **difference of squares** formula $x^2 - y^2 = (x-y)(x+y)$:

$$x^2 - 1 = x^2 - 1^2 = (x-1)(x+1).$$

Therefore,

$$\frac{x^2 - x}{x^2 - 1} = \frac{x(x-1)}{(x-1)(x+1)} = \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{x}{x+1}.$$

Simplify before evaluating

Simplify, if you can, before evaluating.

For example, if we need to evaluate $\frac{x^2 - x}{x^2 - 1}$ at $x = 14$,

then a straightforward evaluation is cumbersome:

$$\left.\frac{x^2 - x}{x^2 - 1}\right|_{x=14} = \frac{14^2 - 14}{14^2 - 1} = \frac{196 - 14}{196 - 1} = \frac{182}{195},$$

but it gets easier if we simplify first: $\frac{x^2 - x}{x^2 - 1} = \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{x}{x+1}$,

then evaluate: $\left.\frac{x}{x+1}\right|_{x=14} = \frac{14}{14+1} = \frac{14}{15}$. Is $\frac{182}{195} = \frac{14}{15}$?

Yes, because $\frac{182}{195} = \frac{14 \cdot \cancel{13}}{\cancel{13} \cdot 15} = \frac{14}{15}$. Observe that $x - 1\big|_{x=14} = 14 - 1 = 13$.

Something may go wrong

Evaluate the same expression $\dfrac{x^2-x}{x^2-1}$ at $x=1$.

Using the same simplification $\dfrac{x^2-x}{x^2-1} = \dfrac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \dfrac{x}{x+1}$, we get

$$\left.\frac{x}{x+1}\right|_{x=1} = \frac{1}{1+1} = \frac{1}{2}$$

Using the original expression $\dfrac{x^2-x}{x^2-1}$, we get

$$\left.\frac{x^2-x}{x^2-1}\right|_{x=1} = \frac{1^2-1}{1^2-1} = \frac{0}{0}$$ Oops! Division by 0 is impossible!

$\left.\dfrac{x^2-x}{x^2-1}\right|_{x=1}$ is not defined, while $\left.\dfrac{x}{x+1}\right|_{x=1} = \dfrac{1}{2}$, although $\dfrac{x^2-x}{x^2-1} = \dfrac{x}{x+1}$!

Why this happens and how to avoid

How could this happen? Let us analyse our calculations:

$$\left.\frac{x^2-x}{x^2-1}\right|_{x=1} = \left.\frac{x(x-1)}{(x-1)(x+1)}\right|_{x=1} = \frac{(1)(1-1)}{(1-1)(1+1)} = \frac{1\cdot 0}{0\cdot 2}$$

It is OK to cancel out $x-1$ in $\dfrac{x(x-1)}{(x-1)(x+1)}$,

but $x-1\big|_{x=1} = 1-1 = 0$, and cancellation by 0 is impossible!

It is useful and safe to simplify a rational expression $\dfrac{p(x)}{q(x)}$ prior to evaluating at $x=a$, if $q(a)\neq 0$.

Summary

In this lecture, we have learned

- ☑ what a **rational expression** is
- ☑ how to **evaluate** a rational expression at a number
- ☑ when a rational expression is **not** defined
- ☑ how to **substitute** an expression into a rational expression
- ☑ how to cancel a **common factor**
- ☑ how to **simplify** a rational expression