Lecture 10

Operations with Polynomials

Reminder: what is a polynomial?

We learned in Lecture 9 that

A **polynomial** is an expression involving numbers, variables and operations of addition, subtraction and multiplication.

Any polynomial in one variable can be written in the **standard form**

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \,,$$

where x is a variable, n is a non-negative integer, and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are **coefficients** (constants).

The highest power of x is called the **degree** of the polynomial.

In this lecture, we will learn how to **operate** with polynomials.

Addition and subtraction

If we **add** or **subtract** two polynomials, then the resulting expression is again a polynomial.

Example 1. Let $p = 2x^3 - 4x^2 + x - 1$ and $q = x^3 + 3x^2 - 4x + 2$ be two polynomials. Find $p + q$ and $p - q$ and put them in standard form.

Remark. We have given the polynomials the names, p and q. It is common in mathematics to give short names to long expressions.

Solution.

$p + q = \underbrace{(2x^3 - 4x^2 + x - 1)}_{p} + \underbrace{(x^3 + 3x^2 - 4x + 2)}_{q}$ This is the sum. Put it in **standard form**:

$$\underset{\text{combine similar terms}}{=} (2x^3 + x^3) + (-4x^2 + 3x^2) + (x - 4x) + (-1 + 2) = \underbrace{3x^3 - x^2 - 3x + 1}_{\text{standard form}}.$$

Subtraction

Now we calculate $p - q$, where $p = 2x^3 - 4x^2 + x - 1$ and $q = x^3 + 3x^2 - 4x + 2$ as before.

$$p - q = \underbrace{(2x^3 - 4x^2 + x - 1)}_{p} - \underbrace{(x^3 + 3x^2 - 4x + 2)}_{q} =$$

$$2x^3 - 4x^2 + x - 1 - x^3 - 3x^2 + 4x - 2 =$$

$$(2x^3 - x^3) + (-4x^2 - 3x^2) + (x + 4x) + (-1 - 2) =$$

$$x^3 - 7x^2 + 5x - 3.$$

Multiplication

If we multiply two polynomials, then the resulting expression is a polynomial.

Example 1. Let $p = 2x - 1$ and $q = -x^2 + 3x + 4$ be two polynomials.
Find the polynomial pq, put it in standard form and determine its degree.

Solution.

$$pq = (2x - 1)(-x^2 + 3x + 4) =$$

$$2x(-x^2) + (2x)(3x) + (2x) \cdot 4 + (-1)(-x^2) + (-1)(3x) + (-1) \cdot 4 =$$

$$-2x^3 + 6x^2 + 8x + x^2 - 3x - 4 = -2x^3 + 7x^2 + 5x - 4\,.$$

Therefore, $pq = -2x^3 + 7x^2 + 5x - 4$. The degree of pq is 3.

In general, if p and q are polynomials of degree n and m respectively,
then their product pq has the degree $n + m$.

That is, when we multiply polynomials, their degrees are added.

Short multiplication formulas

$$(x+y)^2 = x^2 + 2xy + y^2 \quad \text{for any } x \text{ and } y$$

Indeed,

$(x+y)^2 = (x+y)(x+y) = x \cdot x + \underbrace{x \cdot y + y \cdot x}_{2xy} + y \cdot y = x^2 + 2xy + y^2\,.$

This formula will save you an enormous amount of time. It's worth memorizing!

Examples. $(x+3)^2 = x^2 + 2x \cdot 3 + 3^2 = x^2 + 6x + 9\,.$
$(3a+4b)^2 = (3a)^2 + 2(3a)\cdot(4b) + (4b)^2 = 9a^2 + 24ab + 16b^2\,.$

A similar formula for the difference:

$$(x-y)^2 = x^2 - 2xy + y^2 \quad \text{for any } x \text{ and } y$$

Examples. $(xz-5)^2 = (xz)^2 - 2(xz)\cdot 5 + 5^2 = x^2z^2 - 10xz + 25\,.$
$(2a-1)^2 = (2a)^2 - 2(2a)\cdot(1) + 1^2 = 4a^2 - 4a + 1\,.$

Factoring

To **factor** a polynomial means to present the polynomial as a product of non-constant polynomials.

For example, we factor $3x^2 + x$ as follows:

$$3x^2 + x = x(3x+1)\,.$$

Factoring is opposite to multiplication:

$$x(3x+1) \underset{\text{factoring}}{\overset{\text{multiplication}}{\rightleftarrows}} 3x^2 + x$$

Multiplication of polynomials is straightforward:
given two polynomials, you can **always** multiply them.

Factoring may be **difficult** or **impossible**.

Factoring out a monomial

Example 1. Factor the polynomial $4x^3 + 5x^2$.

Solution. The monomials $4x^3$ and $5x^2$ have the common factor of x^2:

$$4x^3 = x^2 \cdot 4x \quad \text{and} \quad 5x^2 = x^2 \cdot 5 \,.$$

We factor out x^2:

$$4x^3 + 5x^2 = x^2 \cdot 4x + x^2 \cdot 5 = x^2(4x + 5) \,.$$

Example 2. Factor the polynomial $10x^3 + 6x^2 - 4x$.

Solution. The monomials $10x^3$, $6x^2$ and $4x$ have the common factor of $2x$:

$$10x^3 = 2x \cdot 5x^2 \,, \quad 6x^2 = 2x \cdot 3x \,, \quad \text{and } 4x = 2x \cdot 2 \,.$$

We factor out $2x$:

$$10x^3 + 6x^2 - 4x = 2x \cdot 5x^2 + 2x \cdot 3x - 2x \cdot 2 = 2x(5x^2 + 3x - 2) \,.$$

Remark. As we will learn later, the polynomial $5x^2 + 3x - 2$ can be factored further:

$$5x^2 + 3x - 2 = (5x - 2)(x + 1) \,.$$

Difference of squares

$$x^2 - y^2 = (x - y)(x + y) \quad \text{for any } x \text{ and } y$$

Indeed,

$(x + y)(x - y) = x \cdot x + x(-y) + y \cdot x + y(-y) = x^2 - xy + xy - y^2 = x^2 - y^2$.

Example 1. Factor $x^2 - 1$.

Solution. $x^2 - 1 = x^2 - 1^2 = (x - 1)(x + 1)$.

Example 2. Factor $4 - a^2$.

Solution. $4 - a^2 = 2^2 - a^2 = (2 - a)(2 + a)$.

Example 3. Factor $9x^4 - y^6$.

Solution. $9x^4 - y^6 = (3x^2)^2 - (y^3)^2 = (3x^2 - y^3)(3x^2 + y^3)$.

Example 4. Factor $x^4 - 1$.

Solution. $x^4 - 1 = (x^2)^2 - 1^2 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$.

Evaluation of a polynomial at a number

Let p be a polynomial in a single variable x. As any expression, p may be evaluated at a number.

"Evaluating p at 2", say, means substituting 2 for every occurrence of x in p. This gives a number, the **value of** p **at** 2, which we denote by $p(2)$.
The polynomial p itself can then also be denoted by $p(x)$.

Example 1. Let $p(x) = 3x^2 - x + 4$. Find $p(0)$, $p(1)$, $p(-2)$.

Solution. We have to **evaluate** the polynomial $p(x)$ at numbers $0, 1, -2$.
For this, we **substitute** (plug in) $x = 0$, $x = 1$, and $x = -2$, into $p(x)$.

$$p(0) = p(x)\Big|_{x=0} = 3x^2 - x + 4\Big|_{x=0} = 3 \cdot 0^2 - 0 + 4 = 4.$$

$$p(1) = p(x)\Big|_{x=1} = 3x^2 - x + 4\Big|_{x=1} = 3 \cdot 1^2 - 1 + 4 = 3 - 1 + 4 = 6.$$

$$p(-2) = p(x)\Big|_{x=-2} = 3x^2 - x + 4\Big|_{x=-2} = 3 \cdot (-2)^2 - (-2) + 4 = 3 \cdot 4 + 2 + 4 = 12 + 2 + 4 = 18.$$

Evaluation of a polynomial at a number

Remark. The polynomial $p(x) = 3x^2 - x + 4$ describes the following algorithm:

$$x \xrightarrow{\text{multiply by } x} x^2 \xrightarrow{\text{multiply by } 3} 3x^2 \xrightarrow{\text{subtract } x} 3x^2 - x \xrightarrow{\text{add } 4} 3x^2 - x + 4$$

Evaluation of $p(x)$ at a given number, say 1, is plugging $x = 1$ into the algorithm:

$$1 \xrightarrow{\text{multiply by } 1} 1^2 \xrightarrow{\text{multiply by } 3} 3 \cdot 1^2 \xrightarrow{\text{subtract } 1} 3 \cdot 1^2 - 1 \xrightarrow{\text{add } 4} \underbrace{3 \cdot 1^2 - 1 + 4}_{6}$$

Note that $p(x)$ does **not** mean $p \cdot (x)$. If p is a polynomial in the variable x, then $p(x)$ is just another notation for p. We do **not** mean to multiply p by x!

Substitution

Example 1. Let $p(x) = -x^2 + 3x$. Find $p(a)$, $p(a-1)$, $p(a^2)$.

Remark. We have to substitute $x = a$, $x = a - 1$, $x = a^2$ into $p(x)$.

This procedure is called a **substitution**. Substitution is like **evaluation**, but instead of a number, we plug in an algebraic **expression**.

Solution. $p(a) = -x^2 + 3x\Big|_{x=a} = -a^2 + 3a.$

$$p(a-1) = -x^2 + 3x\Big|_{x=a-1} = -(a-1)^2 + 3(a-1)$$
$$= -(a^2 - 2a + 1) + 3(a-1)$$
$$= -a^2 + 2a - 1 + 3a - 3 = -a^2 + 5a - 4.$$

$$p(a^2) = -x^2 + 3x\Big|_{x=a^2} = -(a^2)^2 + 3a^2 = -a^4 + 3a^2.$$

Summary

In this lecture, we have learned

- ☑ how to **add** and **subtract** polynomials
- ☑ how to **multiply** polynomials
- ☑ formulas for **short multiplication**: $(x+y)^2 = x^2 + 2xy + y^2$
  $(x-y)^2 = x^2 - 2xy + y^2$
- ☑ how to **factor out** monomials
- ☑ the formula for **difference of squares**: $x^2 - y^2 = (x-y)(x+y)$
- ☑ how to **evaluate** a polynomial at a number
- ☑ how to **substitute** an expression into a polynomial