Lecture 9

Polynomials

What is a polynomial?

A **polynomial expression** is an expression that involves only numbers, variables and the operations of addition, subtraction and multiplication. Division by a number is allowed (because it is a multiplication by the reciprocal number), but division by an expression which contains a variable is not allowed.

Example 1. $x^2 + x$ is a polynomial expression. It involves a variable x and operations of multiplication and addition: $x^2 + x = x \cdot x + x$.

Example 2. $x(x+1)$ is a polynomial expression. It involves a variable x and operations of addition and multiplication.

The polynomial expressions in Examples 1 and 2 are **equal:** $x^2 + x = x(x+1)$.

We say that they represent the **same polynomial**.

Example 3. 1 is a polynomial expression because it involves a single number 1, and neither variables nor operations.
In general, any **constant** (number) is a polynomial.

Example 4. x is a polynomial in **one** variable x.

What is a monomial?

Example. $2x^3$ is a polynomial in **one** variable x, because it is represented by an expression involving the constant 2, the variable x, and three operations (multiplications):

$$2x^3 = 2 \cdot x \cdot x \cdot x.$$

An expression like ax^n, where a is a constant and x^n is a variable x raised to a **non-negative** power n is called a **monomial**.

A monomial is a polynomial with **neither** addition **nor** subtraction involved.

Examples of monomials: $4x$, $-5x^2$, $\frac{2}{5}x^3$.

Any constant is a monomial. For example, 3 is a monomial, since $3 = 3 \cdot \underbrace{x^0}_{1}$.

More examples of polynomials

Example 4. $-2x^3 + x^2 + 4x - 1$ is a polynomial in **one** variable x. It is the sum of four monomials.

• **A sum of several monomials is a polynomial.**

Example 5. $x(x(-2x+1)+4)-1$ is a polynomial in **one** variable x.

Let us clear parentheses in this polynomial:

$$x(x(-2x+1)+4)-1 = x(x(-2x)+x\cdot 1+4)-1$$
$$= x(-2x^2+x+4)-1 = -2x^3+x^2+4x-1.$$

We see that the polynomial $x(x(-2x+1)+4)-1$ is actually the polynomial from Example 4:

$$x(x(-2x+1)+4)-1 = -2x^3+x^2+4x-1\,.$$

A polynomial may be presented by different polynomial expressions.

Polynomials in several variables

Example 1. $3xy^2$ is a polynomial in **two** variables x, y. It is a **monomial** (the product of a constant and powers of variables).

Example 2. $3xy(3x+1)(4y-2)+x-1$ is a polynomial in **two** variables x, y.

Example 3. $x+2y^2+z^3-xy-3xz^7$ is a polynomial in three variables x, y, z. It is the sum of five monomials.

Polynomial or not?

Example 1. $x + \frac{1}{x}$ is **not** a polynomial. This expression involves **division** by a variable. Division by variables is **not** allowed in polynomials.

Example 2. $\frac{x+1}{2}$ is a polynomial. Division by 2 is actually multiplication by $\frac{1}{2}$:

$$\frac{x+1}{2} = \frac{1}{2}(x+1) = \frac{1}{2}x + \frac{1}{2}.$$

Division by any non-zero number is a multiplication by its reciprocal.

Example 3. $x^{-2} + 3x - 1$ is **not** a polynomial.
x^{-2} can't show up in a polynomial, because
a polynomial can't contain a variable with negative exponent.

Simplifying polynomial expressions

Expressions representing polynomials may be simplified.

Example 1. Clear parentheses in the expression $x^3(2x-1)$.

Solution. We distribute x^3 to clear parentheses:

$x^3(2x-1) = x^3 \cdot (2x) + x^3(-1) = 2x^3 \cdot x - x^3 = 2x^4 - x^3.$

Example 2. Clear parentheses and combine similar terms in the expression $(x+2)(3x-1)$.

Solution. We use distributivity to clear parentheses:

$(x+2)(3x-1) = x(3x) + x(-1) + 2(3x) + 2(-1) = 3x^2 - x + 6x - 2 = 3x^2 + 5x - 2.$

Example 3. Clear parentheses: $(-2x^3 + x - 4)(5x+1)$.

Solution. We distribute, and then combine similar terms:

$(-2x^3 + x - 4)(5x+1) = -2x^3 \cdot 5x + (-2x^3) \cdot 1 + x \cdot 5x + x \cdot 1 + (-4)(5x) + (-4) \cdot 1$
$= -10x^4 - 2x^3 + 5x^2 + x - 20x - 4 = -10x^4 - 2x^3 + 5x^2 - 19x - 4.$

The standard form of a polynomial in one variable

No matter how a polynomial in one variable is written, one can use commutativity, associativity and distributivity to put it in **standard form**:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where x is the variable, n is a non-negative number, and $a_0,\ a_1,\ a_2, \ldots, a_{n-1}, a_n$ are constants.

Scared by this "letter monster"? Let us take it apart, to see what it is made of.

As we know, x is a **variable**.
The letter n stands for the non-negative (positive or zero) number, which is the highest power of x in the expression. It is called the **degree** of the polynomial.

The letters $a_0,\ a_1,\ a_2, \ldots, a_{n-1}, a_n$ stand for constants (numbers).
They are called the **coefficients** of the polynomial.

The word "polynomial" means "many parts".

Taming the monster

Let us see how to put a polynomial in **standard form**:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \,.$$

Example 1. Put the polynomial $(4x-1)(2x+3)$ in standard form.

Solution. We distribute, and combine similar terms:

$$(4x-1)(2x+3) = 4x \cdot 2x + 4x \cdot 3 - 1 \cdot 2x - 1 \cdot 3 = 8x^2 + 12x - 2x - 3 = 8x^2 + 10x - 3.$$

The resulting expression, $8x^2 + 10x - 3$, is a polynomial in **standard form**.

Indeed, the highest power of x is $n = 2$. And the long expression

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is reduced in this case to

$$a_2 x^2 + a_1 x + a_0$$

with $n = 2$, $a_2 = 8$, $a_1 = 10$ and $a_0 = -3$: $\underbrace{8}_{a_2} x^2 + \underbrace{10}_{a_1} x + \underbrace{-3}_{a_0}.$

Polynomials in standard form

Example 1. Write the polynomial $2x + 3x^4 - x^3 + 1$ in **standard form**, identify the **coefficients**, and determine the **degree** of the polynomial.

Solution. Rearrange the monomials in **descending** order of exponents:

$$2x + 3x^4 - x^3 + 1 = 3x^4 - x^3 + 2x^1 + 1x^0.$$

The standard form is $3x^4 - x^3 + 0 \cdot x^2 + 2x + 1$. The degree is $n = 4$.
The coefficients are $a_4 = 3,\ a_3 = -1,\ a_2 = 0,\ a_1 = 2,\ a_0 = 1$.
Observe that the term containing x^2 is included with thith the coefficient 0.

Example 2. What is the degree of the polynomial 1?

Solution. As we know, any constant is a polynomial. Actually, it is a monomial. In our case, $1 = 1x^0$. The degree is the **highest** power of x, which is 0.

Answer: the degree of the polynomial 1 is **zero**.

Remark. Any constant is a polynomial of degree **zero**.

Summary

In this lecture, we have learned

- ☑ what **polynomial expressions** are
- ☑ what **polynomials** are
- ☑ what **monomials** are
- ☑ that polynomials may be in **one** or **several** variables
- ☑ how to **simplify** polynomial expressions
- ☑ that the **standard form** of a polynomial is
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$
- ☑ how to identify the **degree** of a polynomial
- ☑ how to identify the **coefficients** of a polynomial
- ☑ how to bring a polynomial to the standard form